# Relations

**Def:** A context free grammar is a **simple precedence grammar** if

1. all right hand sides are unique,
2. there are no empty right hand sides, and
3. at most one simple precedence relation holds between pairs of symbols in the grammar.

The first two conditions are easily verifiable by simply inspecting the grammar. The third condition requires a little work. There are three simple precedence relations that hold between pairs of symbols in a grammar: less than (<), equals (=), and greater than (>). They are defined as

**Def:** $V_i = V_j$ if $\exists$ a right hand side of a production of the form $\alpha V_i V_j \beta$ where $V_i, V_j \in V$ and $\alpha, \beta \in V^*$.

**Def:** $V_i < V_j$ if $\exists$ a right hand side of a production of the form $\alpha V_i V_k \beta$ and $V_k \Rightarrow^+ V_j \delta$ where $V_k \in V_n$, $V_i, V_j \in V$, and $\alpha, \beta, \delta \in V^*$.

**Def:** $V_i > V_j$ if $\exists$ a right hand side of a production of the form $\alpha V_k V_j \beta$ and $V_k \Rightarrow^+ \delta V_i$
or
if $\exists$ a right hand side of a production of the form $\alpha V_k V_l \beta$ and $V_k \Rightarrow^+ \delta V_i$ and $V_l \Rightarrow^+ V_j \gamma$ where $V_k, V_l \in V_n$, $V_i, V_j \in V$, and $\alpha, \beta, \delta, \gamma \in V^*$.

Let's also define two binary relations that will be helpful in computing the simple precedence relations later.

**Def:** $V_i \; h \; V_j$ holds if $V_i \rightarrow V_j \alpha \in P$ where $V_i \in V_n$, $V_j \in V$ and $\alpha \in V^*$.

h is called the *immediate head symbol* relation. If $V_i \; h \; V_j$ holds, then we say that $V_i$ has an immediate head symbol $V_j$. Furthermore let the binary relation H be defined as

**Def:** $V_i \; H \; V_j$ holds if $V_i \Rightarrow^+ V_j \; \alpha$ where $V_i \in V_n$, $V_j \in V$ and $\alpha \in V^*$.

H is called the *head symbol* relation. If $V_i \; H \; V_j$ holds, then we say that $V_i$ has a head symbol $V_j$. Note that H is the transitive closure of h.

$$H = h^{\wedge}$$

In a similar fashion, we can define

**Def:** $V_i \; t \; V_j$ holds if $V_i \rightarrow \alpha \; V_j \in P$ where $V_i \in V_n$, $V_j \in V$ and $\alpha \in V^*$.

t is called the *immediate tail symbol* relation. If $V_i$ t $V_j$ holds, then we say that $V_i$ has an immediate tail symbol $V_j$. Furthermore let the binary relation T be defined as

**Def:** $V_i$ T $V_j$ holds if $V_i \Rightarrow^+ \alpha V_j$ where $V_i \in V_n$, $V_j \in V$ and $\alpha \in V^*$.

T is called the *tail symbol* relation. If $V_i$ T $V_j$ holds, then we say that $V_i$ has a tail symbol $V_j$. Note that T is the transitive closure of t.

$$T = t^{\wedge}$$

Next let us define a binary relation for the **equals** simple precedence relation by

**Def:** $V_i$ E $V_j$ holds if $\exists$ a right hand side of a production of the form $\alpha V_i V_j \beta$ where $V_i, V_j \in V$ and $\alpha, \beta \in V^*$.

If $V_i$ E $V_j$ holds, then we say that $V_i$ can be next to (equals) $V_j$ in some right hand side of a production in the grammar. Note that by inspecting the grammar, we can identify all E relations that hold. The **less than** simple precedence relation can be identified as

**Def:** $V_i$ L $V_j$ holds if $\exists$ a right hand side of a production of the form $\alpha V_i V_k \beta$ and $V_k \Rightarrow^+ V_j \delta$ where $V_k \in V_n$, $V_i, V_j \in V$, and $\alpha, \beta, \delta \in V^*$.

If $V_i$ L $V_j$ holds, then we say that $V_i$ is less than $V_j$ and that $V_j$ starts a phrase in a sentential form. Let's investigate the L relation a little further. Note that I can write the binary relation in the following form:

$V_i$ L $V_j$ holds if $V_i$ E $V_k$ and $V_k$ H $V_j$

This is the same definition with the binary relations substituted for the English narrative. If the E and H are binary relations that are represented by Boolean matrices E and H, respectively, then we have

$$\mathbf{L = E\ H}$$

via relational composition. Hence, if the equals relations are represented as a Boolean matrix and the head symbols are also represented as a Boolean matrix and if the row and columns are ordered identically, then the less than relations can be computed by multiplying the E matrix with the H matrix. The **greater than** simple precedence relation can also be defined using binary relations as

**Def:** $V_i$ G $V_j$ holds if $\exists$ a right hand side of a production of the form $\alpha V_k V_j \beta$ and $V_k \Rightarrow^+ \delta V_i$
or
if $\exists$ a right hand side of a production of the form $\alpha V_k V_l \beta$ and $V_k \Rightarrow^+ \delta V_i$ and $V_l \Rightarrow^+ V_j \gamma$ where $V_k, V_l \in V_n$, $V_i, V_j \in V$, and $\alpha, \beta, \delta, \gamma \in V^*$.

Let's separate the definition into its two parts, namely a $G_1$ and a $G_2$.

**Def:** $V_i \; G_1 \; V_j$ holds if $\exists$ a right hand side of a production of the form $\alpha \, V_k \, V_j \, \beta$ and $V_k \Rightarrow^+ \delta \, V_i$

and

**Def:** $V_i \; G_2 \; V_j$ holds if $\exists$ a right hand side of a production of the form $\alpha \, V_k \, V_l \, \beta$ and $V_k \Rightarrow^+ \delta \, V_i$ and $V_l \Rightarrow^+ V_j \, \gamma$ where $V_k, V_l \in V_n$, $V_i, V_j \in V$, and $\alpha, \beta, \delta, \gamma \in V^*$.

If $V_i \; G_1 \; V_j$ holds, then we say that $V_i$ is greater than $V_j$ and that $V_i$ ends a phrase in a sentential form. If $V_i \; G_2 \; V_j$ holds, then we say that $V_i$ is greater than $V_j$ and that $V_i$ ends a phrase and $V_j$ starts a phrase in a sentential form. We can also write the binary relations for $G_1$ and $G_2$ as

$V_i \; G_1 \; V_j$ holds if $V_k \; E \; V_j$ and $V_k \; T \; V_i$

$V_i \; G_2 \; V_j$ holds if $V_k \; E \; V_l$ and $V_k \; T \; V_i$ and $V_l \; H \; V_j$

Concentrating on $G_1$ for the moment, we note that the above form is not in the correct form for relational composition. However, by again re-writing the above definition we can get the definition into the correct form for relational composition, namely

$V_i \; G_1 \; V_j$ holds if $V_k \; E \; V_j$ and $V_i \; T^T \; V_k$

Again re-writing we have

$V_i \; G_1 \; V_j$ holds if $V_i \; T^T \; V_k$ and $V_k \; E \; V_j$

which is in the correct form for relational composition. If we have Boolean matrices T and E that represent the binary relations T and E, respectively, then we have

$$G_1 = T^T \; E$$

In a similar fashion let's take a look at $G_2$. From

$V_i \; G_2 \; V_j$ holds if $V_k \; E \; V_l$ and $V_k \; T \; V_i$ and $V_l \; H \; V_j$

we can make the following re-write

$V_i \; G_2 \; V_j$ holds if $V_i \; T^T \; V_k$ and $V_k \; E \; V_l$ and $V_l \; H \; V_j$

which is in the correct form for relational composition. Hence, if we have Boolean matrices T, E, and H that represent the binary relations T, E and H, respectively, then we have

$$G_2 = T^T \; E \; H$$

or by substituting L for E H we have

$$G_2 = T^T \; L$$

Combining the $G_1$ and $G_2$ to yield G, we have

$$G = G_1 + G_2$$

or

$$G = T^T E + T^T L$$

or factoring $T^T$ we get

$$\mathbf{G = T^T ( E + L )}$$

**Review:**

Computationally, we have

1. Identify **h** by looking at the grammar
2. Compute **H** by taking the transitive closure of h
3. Identify **t** by looking at the grammar
4. Compute **T** by taking the transitive closure of t
5. Identify **E** by looking at the grammar
6. Compute **L** by multiplying **E H**
7. Compute **G** by multiplying $\mathbf{T^T ( E + L )}$