Convergence of a Transformation on a Weighted Graph

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Abstract

Kramer and Bruckner defined the following transformation on a weighted graph on n vertices. The transformation replaces the weight at every vertex by either the minimum or the maximum of the weights in its closed neighborhood, the choice being that extremum which is closer in value to the original weight. They showed that the system always converges after a finite number of iterations of the transform. We consider here the question of how fast this convergence is, and show that $O(n^2)$ iterations suffice. We conjecture though that $n-2$ iterations suffice, and verify this for some special graphs. The convergence rates of more general transforms are also considered.

1 Introduction

Let G be a graph with values assigned to the vertices. With a view to an algorithm to sharpen pictures, Kramer and Bruckner [2] introduced the following transform of the values. Under the transform, each vertex v with value $w(v)$ is given a new value $w'(v)$ according to the *nearest-extremum* rule: if $m(v)$ and $M(v)$ are the minimum and maximum value of v and its neighbors, then

NE: $w'(v) :=$ nearer of $m(v)$ and $M(v)$ to $w(v)$.

We assume some consistent rule for ties, say round-up. We shall use the term neighborhood to include the vertex itself.

Kramer and Bruckner [2] considered repeated applications of the transform or *rounds*. They showed that this procedure will always *converge* after a finite number of iterations, i.e. a stage will be reached where every vertex is a *local extremum* in that it has the largest or smallest value in its neighborhood. Another proof was given by Johnson [1].

We show here a quadratic (in the number of vertices) upper bound on how many rounds the procedure takes to converge. We also give linear upper bounds for some special graphs, and investigate how many rounds the procedure can last under related transforms.

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2 Examples

We define first a useful concept: a vertex is said to *die* if its value changes to its final value. By extension we speak of alive and dead vertices.

Two different constraints are suggested by the nearest-extremum rule. These are the *forced-move* constraint,

FE: $w'(v) \in \{m(v), M(v)\}$,

and the *stay* constraint,

S: **if** $w(v) \in \{m(v), M(v)\}$, **then** $w'(v) = w(v)$.

We will call any rule consistent with these an **FES** rule. Clearly the nearest-extremum rule is such a rule.

As a trivial example, consider the complete graph. Under the forced-move constraint, after the first round every vertex has value either the minimum or maximum initial value. Under the stay constraint, these vertices are therefore dead, and thus **FES** forces convergence in one round.

Another example is the complete bipartite graph. The following is easily established:

Example 1 *The process on the complete bipartite graph can last at most* 2 *rounds under* **NE***, and at most* 3 *rounds under* **FES***, and these are attainable.*

Another example is the path:

Example 2 *The process on the path with n vertices can last at most $n-2$ rounds under the* **FES** *or* **NE** *rule, and this is attainable under either rule.*

Proof: Consider first any vertex x which is a local maximum. We claim that this dominance is permanent, and thus that x is dead. For let y be any neighbor of x. For y to increase it must be adjacent to a vertex smaller than it. But then it increases to x's value, as x is its only other neighbor.

So we need only examine the case where the values form an ascending (say) sequence. We claim that a vertex must die. Work upwards and look for the first vertex that decreases. Such a vertex becomes a local minimum and thus dies. But if every vertex increases then the second largest gets the Maximum value and dies.

As the path with values consecutively 1 through n lasts $n-2$ rounds, this completes the proof. QED

This result suggests the following definition: a graph is *obliging* if for any choice of weights one vertex must die under the nearest-extremum transform. We believe that this behavior is typical:

Conjecture 1 *The nearest-extremum rule guarantees convergence in* $n-2$ *rounds. Moreover, every graph is obliging.*

We conclude this section by establishing that **FES** guarantees convergence. Kramer and Bruckner [2] and Johnson [1] used a more complicated argument to show that **NE** guaranteed convergence.

Lemma 1 *Consider a rule that replaces the value at every vertex by one of the values in its neighborhood and which always makes at least one move each round. If the rule obeys*

$$\mathbf{FES}^{\mathrm{down}}\text{: } \mathbf{if}\ w'(v) < w(v)\ \textit{or}\ w(v) = m(v)\ \mathbf{then}\ w'(v) = m(v),$$

then the process converges.

Proof: Clearly there is a finite number of arrangements of values for the graph.

Suppose the same arrangement $\mathcal{P}$ occurs twice. Consider the sequence of arrangements between two occurrences of $\mathcal{P}$. Consider in each round, the vertices whose values change. Then let v be a vertex whose new value A is smallest over the whole cycle.

Say in round i v changes from B to A. Eventually v must move back to B; thus $B > A$. Hence in round i, v becomes $m(v)$ the minimum in its neighborhood. Now, v will stay at this value until one of its neighbors decreases. But by the definition of v and A, this will never happen, which is a contradiction! QED

3 Quadratic Upper Bounds for NE

In this section we provide a quadratic bound on the time to convergence under the nearest-extremum rule. We do this by defining an event called a super jump, which results in the death of a vertex. Then we prove a linear upper bound on the time before a super jump. This, together with some other observations, yields the desired bound.

We start with a simple technical lemma.

Lemma 2 *We may assume that no value is the average of two others.*

Proof: Consider a given system with positive values. Let ε be much smaller than the smallest value there. Then replace the value w with the value $w - \varepsilon w^2$. It is easily checked that the evolution is unchanged. (Recall that we are assuming round-up for ties.) QED

3.1 The History Approach

Consider a jump J in round t of size D. Say v goes *up* from zero. Then after round $t-1$, v was adjacent to two vertices: x with value D and y with value $-E$ where $D < E$ (by the above lemma). Vertex v tried to become a local extremum in round $t-1$. If v maximized then x went up from at most zero. If v minimized then y went down from at least zero. So, by induction, a jump J has a *history of jumps of at least equal size.*

3.2 The History of a Supreme Jump

We focus in on jumps up to the highest ever jumped up to: we call such events *super up-jumps*. Further, we define a *supreme up-jump* as a super up-jump whose immediate predecessor was not a super up-jump.

So assume that J is a supreme up-jump. Then x did not move up in round $t-1$, so v took the minimum value in its neighborhood. Thus y went down from at least zero to $-E$. The precursor of y's jump also cannot have been an up-jump (it would have to have been to at least E), so it too was a down-jump from at least zero to $-E$. Thus in the history of J, all the jumps are down-jumps to $-E$.

Now for y to drop to $-E$ in round $t-1$, it must then have been adjacent to a vertex with value at least E and hence greater than D. Working backwards it follows that each jump in the history of J is by a vertex with a neighbor with value greater than D.

By the supremacy of J, vertices with value greater than D have always had value greater then D. This means that the vertices in the history have not moved up before their historical drop. Thus they are distinct. Furthermore, if both a supreme up- and a supreme down-jump occur, a vertex cannot participate in the histories of both supreme jumps.

Putting all this together we get the following:

Lemma 3 *Let* t^{up} *(resp.* t^{down}*) denote the round when the (last) supreme up-jump (down-jump) occurs. If both up- and down-jumps occur, then*

$$t^{\text{up}} + t^{\text{down}} \leq l + 1,$$

where l *denotes the number of live vertices at the start. If only up-jumps (down-jumps) occur, then* $t^{\text{up}} = 1$ *(*$t^{\text{down}} = 1$*).*

Note that the history of a super up-jump which is not a supreme jump consists of down-jumps, followed by a supreme up-jump and zero or more super up-jumps.

3.3 Supreme Jumps and Two Deaths

It is immediate that the vertex making a super up-jump will forever be a local maximum and thus dies. Thus Lemma 3 implies a quadratic bound.

But we can extract more. In fact, either we can write off two vertices, or something else nice happens. In bounding the time to convergence, clearly we may assume that the dead vertices do not form a cut. Also we may assume that there is a unique global maximum vertex m.

Let M, S and T denote the highest, second-highest and third-highest initial value. There are three possibilities for the supreme up-jump J:

1. *J is to M:* Then it occurs immediately—for otherwise the vertices in the history would be adjacent to vertices with value bigger than M. Further we may immediately merge the two vertices with value M.

2. *J is to S:* We will reset the clock after the last super up-jump and merge the dead vertices with value S into m. We claim that if the supreme up-jump does not occur immediately then there will be at least two such dead vertices.

   Suppose otherwise; that means that at the same time as the supreme up-jump by v, the vertex x which it jumps up to moves down. But that means that x is adjacent to m, and thus v was at least S the previous round. Then v must have moved down the previous round, which would require that v be adjacent to m, a contradiction.

3. *J is to at most T:* Then we argue that after the first round we may assume that there is no vertex with value S.

   Let $\mathcal{S}$ denote the vertices with initial value S. Consider a vertex in $\mathcal{S}$ which is not adjacent to m and thus dead. It is easily seen that we may delete edges to other vertices in $\mathcal{S}$, and then merge it into m without affecting the evolution of the system.

So we may assume that all vertices of $\mathcal{S}$ are adjacent to m. They will thus forever drop. We may assume also that $\{m\}$ is not a cut. This means that all of $\mathcal{S}$ is connected to a smaller value by a path which avoids m. This means that no vertex in $\mathcal{S}$ dies at S. But can it die at T? Well, only if all paths out of that component of $\langle \mathcal{S} \rangle$ are blocked by a dead vertex with value T, but this contradicts the assumption that the dead vertices do not form a cut. So after the first round we can change remaining vertices with value S to value T without affecting the subsequent evolution.

Putting all this together, we may assume that one of the following occurs:

- a supreme up-jump occurs immediately,
- the second-highest value is lost immediately, or
- after the last super up-jump occurs we may write off two vertices.

3.4 The Bound

The first two of these situations only serve to establish a linear bound. So the worst that can happen is the third, and indeed that there is exactly one super up- and one super down-jump. Thus we get:

Theorem 1 *Under the nearest-extremum rule* **NE***, the process can last at most* $n^2/12 + O(n)$ *rounds.*

Proof: Let l denote the number of live vertices. By Lemma 3 and the above discussion, we may assume that: either one supreme jump occurs in $(l+1)/3$ rounds, and we may write off two vertices, or both supreme jumps occur in $2(l+1)/3$ rounds, and we may write off four vertices. Hence the claimed result. (We reset the clock whenever one of our desired events occur.) QED

3.5 Irreversible Jumps

Another quadratic bound can be obtained by looking at irreversible jumps. Let vertex v be a vertex making the biggest jump of size Δ in round i. Say v jumps up. Let y be the vertex of smallest value adjacent to v at round i, so that the gap between y and v is greater than Δ. Then y moves down (if it moves at all). If this gap were ever to diminish, one of v or y would have to jump (at least) the full distance. But, by the history argument, this is impossible.

Hence v is constrained to move up forever and y downwards. We have had an *irreversible jump* by v. Thus there must be an irreversible jump at each round. Trivially, each vertex can participate in at most n rounds.

4 NE and Trees

We prove here a slight generalization of Example 2 on paths. We will need the following simple lemma.

Lemma 4 *If a vertex is a local extremum and all its neighbors have at most two, then it is dead.*

Example 3 *Let T be a tree such that any two vertices of degree at least three are distance two or at least distance four apart. Then T is obliging.*

Proof: Say a vertex is "large" if it has degree three or more. Suppose all interior vertices are alive and no vertex dies in round 1.

Let z_1 and z_2 be large vertices which are at least distance four apart. Let them be connected by $z_1 y_1 x_1 x_2 \ldots x_r y_2 z_2$ ($r \geq 1$). For all the x_i to be alive, none of them can be a local extremum, and thus the sequence $\{y_1, x_1, \ldots, x_r, y_2\}$ is strictly monotone. Consider any x_i; say it moves towards z_1's value. Then its neighbor on z_1's side moves towards z_1, and hence y_1 gets the value of z_1. We write $z_2 \to z_1$; this will mean that the vertex adjacent to z_1 on the way to z_2 either takes z_1's value or remains unchanged.

Consider now large z_1 and z_2 which are distance two apart with common neighbor y. If y takes z_1's value then we write $z_2 \to z_1$; if y stays the same we may write that anyway.

Similarly, consider the path from an end-vertex w to a large vertex z; say it has penultimate vertex y. As we are supposing no vertex dies, y can only stay the same ($d(w,z) \leq 2$), or take z's value ($d(w,z) \geq 2$). In any event, we may write $w \to z$.

Now consider the tree with vertices of degree two contracted out, and every edge directed as above. As an arc points away from a vertex of degree one, there must be an internal sink z. This means that for all neighbors y of z in the original graph T, the value of y after round 1 is either the old value of y or of z. So the set

FIGURE OMITTED

Figure 1: Lower bound for **FES**

of values z can see after round 1 is a subset of those it could see before. Thus after round 1, z must be a local extremum. And, by the above lemma, dead! QED

This can surely be extended.

5 More on FES

For **FES**, we do not have an upper bound which is significantly better than the maximum number of configurations. One thing we can say is that the supreme up-jump must have a "history" that ends with a down jump. Thus, if only (say) down moves ever occur, then process lasts at most $n-2$ rounds.

The configuration for the best lower bound we have is shown in Figure 1. At the start, x is the only non-extremum. It changes to 0, then its neighbor to the right becomes 0, and the process ripples to the right until A becomes 0. Then we let B take $+\infty$, C take -10, and A take -10. The value -10 ripples back to the left. Here it releases the value -100 which ripples to the right again. The process takes $3n - \Theta(1)$ steps.

One may also ask what happens under relaxed constraints. Consider for example the conditions given in the hypothesis of Lemma 1. Then we can exhibit a system which lasts $\Theta(n^3)$ rounds. We sketch the construction. Start with a path P with values 1 through m. This system can last $\Theta(m^2)$ rounds, each round being a single down-jump. Then design a "reset gadget" which allows a vertex v adjacent to all of P to change its value from $+\infty$ to 0 and back again. Changing to 0 allows P to return to its initial configuration (we anchor each vertex on the path to a dead vertex with the same value). By adding a linear number of reset gadgets one obtains a system which lasts a cubic number of rounds. However, there is no reason to believe that this is best possible. Indeed, it may be that such relaxed constraints do not ensure polynomial-time convergence.

References

[1] C.R. Johnson, Convergence of a nonlinear sharpening transformation for digital images, *SIAM J. Alg. Disc. Meth.* **6** (1985), 462–465.

[2] H.P. Kramer and J.B. Bruckner, Iterations of a nonlinear transformation for enhancement of digital images, *Pattern Recognition* **7** (1975), 53–58.