Measures of Vulnerability—The Integrity Family

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Abstract

In this paper a schema of graphical parameters is proposed. Based on the parameter integrity introduced by Barefoot, Entringer and Swart, members $\Psi(G)$ of this schema have the general form $\Psi(G) = \min\{\, |S| + \psi(G - S) : S \subset V(G)\,\}$, where $\psi(G)$ is another given graphical parameter. Examples include integrity, mean integrity, connectivity and vertex cover number. General results and bounds for the schema are derived. Also, properties which characterize such parameters are considered.

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1 Introduction

The attempt to quantify the "vulnerability" of a graph or network began with connectivity and the theorems of Menger and Whitney. The problem is receiving much attention nowadays especially in the field of communications networks. Many measures of vulnerability have been proposed and studied: besides the many variations on connectivity, there are deterministic measures such as toughness [9], and various probabilistic "reliability" measures.

Integrity was introduced by Barefoot et al. [3] as such a measure, though it can be studied as a general parameter. The integrity of a graph G, denoted $I(G)$, is given by

$$I(G) = \min_{S \subset V(G)} (|S| + m(G - S)),$$

where $m(H)$ denotes the maximum order (vertex-cardinality) of a component of graph H. It represents, to some extent, a trade-off between the amount of work done to damage the network and how badly the network is damaged. In this we are assuming that the vertices represent the susceptible part of the network.

Recently, Chartrand et al. [7] introduced the *mean integrity* of a graph which they denoted by $J(G)$. This is defined as integrity except that the parameter m is replaced by $\tilde{m}$, where $\tilde{m}(H)$ is the average order of the components, averaged over the vertices; i.e., if the components of H have orders $p_1, \ldots, p_k$ then

$$\tilde{m}(H) = \sum_{i=1}^{k} p_i^2 / p(H),$$

while $J(G) = \min\{\, |S| + \tilde{m}(G - S) : S \subset V(G)\,\}$. Thus, while integrity is integer-valued, mean integrity is not. Some simple results are summarized in the following proposition:

Proposition 1 [3], [7] *For all graphs G:*
a) $\delta(G) + 1 \le \tilde{m}(G) \le m(G)$;
b) $\kappa(G) + 1 \le \delta(G) + 1 \le J(G) \le I(G) \le \alpha(G) + 1$;
c) if $e \in E(G)$ *then* $I(G - e) \le I(G)$ *and similarly with* J;
d) if $v \in E(G)$ *then* $I(G - v) \le I(G)$ *but not similarly with* J.

Here we propose a new class of measures based on these two parameters. We consider first the properties that characterize parameters in this class. Then we examine some of their graphical properties including how they relate to common graph operations. We conclude by considering possible generalizations of the schema. Most of this work appeared in the author's thesis [11].

In this paper we consider only finite undirected simple graphs. We follow the notation of Chartrand and Lesniak [8]. The number of vertices or *order* of a graph G is denoted by $p(G)$, and the number of edges by $q(G)$. The minimum and maximum degrees are denoted by $\delta(G)$ and $\Delta(G)$ respectively, while $\kappa(G)$ denotes the connectivity, $\alpha(G)$ the vertex cover number, and $\chi(G)$ the chromatic number of G. For two graphical parameters σ and τ we will write $\sigma \leq \tau$ if $\sigma(G) \leq \tau(G)$ for all graphs G.

2 The New Class

Let ψ be a graphical parameter. Then we define the graphical parameter $\Psi_\psi(G)$ (or $\Psi(G)$) *induced* by ψ, by

$$\Psi_\psi(G) = \min_{S \subset V(G)} (|S| + \psi(G - S)).$$

Note that we have not, a priori, imposed any properties on the parameter ψ; desirable properties will be considered later. A simple observation is that $\Psi(K_1) = \psi(K_1)$. Indeed, for convenience we define

$$\psi_0 = \Psi(K_1) = \psi(K_1).$$

Note too that for all graphs G, $\Psi(G) \leq p(G) - 1 + \psi_0$. We also define a Ψ-set of G as any set $S \subset V(G)$ for which $|S| + \psi(G - S) = \Psi(G)$.

The obvious prototypes of this class are integrity and mean integrity. Nevertheless, many common parameters may be formulated in this form. Some use what we call a recognizer function. Let $\mathcal{G}$ be a set of graphs which contains K_1. Then the *recognizer function*, $\psi(G)$, for $\mathcal{G}$ is given by $p(G) - 1$ if $G \notin \mathcal{G}$, and by 0 if $G \in \mathcal{G}$. The associated Ψ measures the minimum number of vertices that must be removed from a graph so that the resulting graph is in $\mathcal{G}$.

Examples of this type are connectivity and vertex cover number. In the first, $\mathcal{G}$ consists of all the disconnected graphs together with the trivial graph, while in the second, $\mathcal{G}$ consists of all the empty graphs. Thus κ and α lie within our schema.

We shall denote these recognizers by ψ_κ and ψ_α respectively. Also, the parameter δ is induced by the recognizer ψ_δ for the set of all graphs with isolated vertices. It so happens that δ is induced by itself also. (These examples show that the value $p(G)-1$ in the definition of a recognizer cannot be replaced by a smaller value, but one may use any larger value.)

The following lemma is a direct consequence of the definitions.

Lemma 1 *Let Ψ and Φ be induced by ψ and φ respectively.*
a) If for all graphs G and $e \in E(G)$ it holds that $\psi(G-e) \leq \psi(G)$, then the same is true for Ψ.
b) If for all nontrivial graphs G and $v \in V(G)$ it holds that $\psi(G-v) \leq \psi(G)$, then the same is true for Ψ.
c) If $\psi \leq \varphi$ then $\Psi \leq \Phi$.

None of the converse implications is necessarily true. For example, if ψ is any parameter such that $\psi(K_1) = 1$ and $\psi(G) \geq p(G)$ for all G then $\Psi = p$. Further, the condition on the functions ψ and φ can be phrased in terms of properties of $\mathcal{G}$ and $\mathcal{H}$ if they are recognizers for these sets.

2.1 Characterizations

We now look at properties which characterize parameters of this class.

It is easily seen that if the parameter Ψ is induced by a recognizer for the set $\mathcal{G}$, then for all non-trivial graphs G it holds that $G \in \mathcal{G}$ iff $\forall v \in V(G)\colon (\Psi(G-v) \geq \Psi(G))$. This motivates the following definition. A graph G is said to be *acritical* with respect to a parameter μ iff either G is trivial or $\forall v \in V(G)\colon (\mu(G-v) \geq \mu(G))$. Further, the set of all graphs acritical with respect to μ will be denoted by $\mathrm{acrit}(\mu)$. Thus, if Ψ is induced by a recognizer for $\mathcal{G}$ then $\mathrm{acrit}(\Psi) = \mathcal{G}$.

Also we call a parameter μ *smooth* iff for all non-trivial graphs G and $v \in V(G)$ it holds that $\mu(G-v) \geq \mu(G)-1$. Examples of smooth parameters include δ, κ, I and χ, but not, for instance, Δ. Note that for smooth μ, $\forall S \subset V(G)\colon (|S| + \mu(G-S) \geq \mu(G))$. These concepts enable us to prove the following results.

Lemma 2 *If Ψ is induced by ψ then*
a) Ψ is smooth;

b) $\Psi \leq \psi$*; and*
c) $\forall G \in \mathrm{acrit}(\Psi)\colon (\Psi(G) = \psi(G))$*.*

Proof. To prove (a), let S be a Ψ-set of $G - v$. Then if $T = S \cup \{v\}$ then $\Psi(G) \leq |T| + \psi(G - T) = |S| + 1 + \psi((G - v) - S) = \Psi(G - v) + 1$. Part (b) follows directly from the definition of $\Psi(G)$. To prove (c), let G be a graph with $\Psi(G) < \psi(G)$. Then, let S be a Ψ-set of G; clearly $S \neq \phi$, so let $v \in S$. Then $\Psi(G - v) \leq |S - v| + \psi((G - v) - (S - v)) = |S| - 1 + \psi(G - S) = \Psi(G) - 1$ so that $G \notin \mathrm{acrit}(\Psi)$. QED

We remark that $\Psi(G) = \psi(G)$ is not a sufficient condition for G to be acritical with respect to Ψ. Consider, for example, $G = K_n$ ($n \geq 2$), $\psi = m$ and $\Psi = I$.

Theorem 3 *Consider the following properties of parameters* μ *and* ψ*:*

R1*:* μ *is smooth;*
R2*:* $\mu \leq \psi$*;*
R3*:* $\forall G \in \mathrm{acrit}(\mu)\colon (\mu(G) = \psi(G))$*; and*
R4*:* μ *is integer-valued.*

Then,
a) properties **R1** *and* **R2** *imply that* $\mu \leq \Psi_\psi$*;*
b) properties **R3** *and* **R4** *imply that* $\mu \geq \Psi_\psi$*;*
c) properties **R1**, **R2** *and* **R3** *are necessary and sufficient conditions for an integer-valued* μ *to equal* Ψ_ψ*.*

Proof. To prove (a), let G be a graph and $S \subset V(G)$. Then $|S| + \psi(G - S) \geq |S| + \mu(G - S) \geq \mu(G)$. As (c) follows from parts (a) and (b) and from the preceding lemma, it remains to prove (b). Let G be a graph. If $G \notin \mathrm{acrit}(\mu)$ then there exists a $v \in V(G)$ such that $\mu(G - v) < \mu(G)$, and hence (by μ being integer-valued) such that $\mu(G - v) \leq \mu(G) - 1$. By repeating the process we obtain an $S^* \subset V(G)$ such that $|S^*| + \mu(G - S^*) \leq \mu(G)$ with $G - S^* \in \mathrm{acrit}(\mu)$. Thus $\Psi_\psi(G) \leq |S^*| + \psi(G - S^*) = |S^*| + \mu(G - S^*) \leq \mu(G)$, and the result is proved. QED

For smooth integer-valued μ we have obtained a characterization of the ψ such that $\Psi_\psi = \mu$: ψ must be an upper bound for μ with equality on $\mathrm{acrit}(\mu)$. Without the integer-valued assumption, the situation is more complex. For example, let ψ be a non-trivial recognizer inducing a parameter Ψ, and let $\mu = \frac{1}{2}\Psi$; then properties **R1**, **R2** and **R3** are satisfied but $\mu \neq \Psi$. Nevertheless:

Theorem 4 *For a graphical parameter μ the following are equivalent:*
(1) $\mu = \Psi_\mu$;
(2) there exists a ψ such that $\mu = \Psi_\psi$; and
(3) μ is smooth.

Proof. Trivially (1) implies (2), while Lemma 2a shows that (2) implies (3). To prove that (3) implies (1), assume that μ is smooth. Then by Theorem 3a it follows that $\mu \leq \Psi_\mu$, while $\Psi_\mu \leq \mu$ by Lemma 2b, so that $\Psi_\mu = \mu$. QED

Thus this schema contains many parameters. For example, the chromatic and clique numbers are in this class. This fact has implications; for example, since trivially $m \geq \chi$, it follows that $I \geq \chi$.

For such a representation to be of use in portraying already-known parameters though, one would like ψ to be easier to manipulate, or for general questions to be answerable. We shall consider both options. The following result characterizes the parameters induced by recognizer functions.

Theorem 5 *A smooth parameter μ is induced by a recognizer function iff μ is integer-valued and* $\operatorname{acrit}(\mu) = \{ G : \mu(G) = 0 \}$.

Proof. The "if" part follows by taking ψ to be the recognizer for $\mathcal{G} = \{ G : \mu(G) = 0 \}$, and applying Theorem 3c. (Since μ is smooth and $\mu(K_1) = 0$ it holds that $\mu(G) \leq p(G) - 1$.)

To prove the "only if" part, let μ be induced by a recognizer. Certainly, μ is integer-valued. Now, if $\mu(G) = 0$ then clearly $G \in \operatorname{acrit}(\mu)$. On the other hand, if $\mu(G) > 0$ then there exists a non-empty μ-set S. By letting $v \in S$ and considering $S - v$, it follows that $\mu(G - v) \leq \mu(G) - 1$, and thus $G \notin \operatorname{acrit}(\mu)$. QED

This theorem shows that neither the clique nor the chromatic number is induced by something as simple as a recognizer function. (For both parameters $2K_n$ is acritical.) Indeed, an important feature is computability. One certainly hopes that ψ is easier to compute than Ψ, suggesting polynomial-time parameters for one's

ψ. All the ψ considered so far are polynomial-time and the associated decision problems thus in NP. On the other hand, it is well-known that computing the parameters κ and α is NP-complete. In [10] it is shown that computing I is NP-complete. By a similar reduction from the vertex cover problem by "weighting" the vertices, one may show that $\{ \langle G, k\rangle : J(G) \leq k \}$ is NP-complete.

One may also seek polynomial-time ψ for other smooth parameters. Consider, for example, the chromatic number χ; then it is necessary and sufficient that ψ be an upper bound for χ with equality on the acritical graphs for χ. However, such a ψ must be just as intractable as χ itself. This may be proved by reduction from χ via $\langle G, k\rangle \mapsto \langle 2G, k\rangle$, since $2G$ is always acritical. A similar result is immediate for the clique number.

2.2 Edge Analogues

Many graphical parameters have edge analogues; for example, edge connectivity κ_1, and edge integrity I' [4]. We consider edge analogues for the class and define for any graphical parameter ψ,

$$\Psi'_\psi(G) = \min_{T \subseteq E(G)} (|T| + \psi(G - T)).$$

Thus Ψ'_{ψ_κ} is κ_1, while Ψ'_m corresponds to I' and Ψ'_{ψ_δ} to δ. On the other hand, the edge analogue of the vertex cover number equals the minimum of $p(G)-1$ and $q(G)$. Most of the results given above have analogues for Ψ'. In defining edge-acritical graphs and edge-smooth parameters, one simply considers $G - e$ rather than $G - v$.

Further, one might use an edge analogue of the recognizer function, by defining $\psi'(G)$ to equal the number of edges if $G \notin \mathcal{G}$, and 0 otherwise. Here the associated parameter Ψ' represents the cardinality of the smallest edge-set which may be removed so that the resultant graph lies in $\mathcal{G}$. Another idea is to define a parameter $m'(H)$ denoting the maximum number of edges of a component of H. This, however, does not yield anything new as for all nonempty graphs G it holds that $\Psi'_{m'}(G)$ is equal to the integrity of the line graph of G. (Thanks to L. Beineke and M. Lipman for pointing this out.)

Now, it is well-known that $\kappa \leq \kappa_1$, while it was observed in [4] that $I \leq I'$. In fact, one may easily establish a similar relationship between Ψ and Ψ'.

Lemma 6 *For a parameter ψ the following are successively weaker:*
(1) $\forall G\colon \forall v \in V(G)\colon (\psi(G-v) \leq \psi(G))$;
(2) $\forall G\colon \forall e \in E(G)\colon \exists v \in V(G)\colon (\psi(G-v) \leq \psi(G-e))$;
(3) $\Psi \leq \Psi'$.

The parameter m satisfies (1) while ψ_κ satisfies (2).

3 Results

We derive some general results including ones for graphical combinations, and when ψ obeys $\psi \geq \Delta + \psi_0$. Throughout this section we will assume that parameter Ψ is induced by parameter ψ.

3.1 Graphical Combinations

We consider the disjoint union ($\cup$) and the join ($+$) of two graphs.

Theorem 7 *Let $G \circ H$ denote $G \cup H$ or $G + H$. Then the following are each sufficient for it to hold that $\forall G, H\colon (\Psi(G \circ H) \leq \Psi(G) + \Psi(H))$:*
(1) $\forall G, H\colon (\psi(G \circ H) \leq \psi(G) + \psi(H))$;
(2) ψ is integer-valued and $\forall G, H \in \mathrm{acrit}(\Psi)\colon (\psi(G \circ H) \leq \psi(G) + \psi(H))$;
(3) Ψ is induced by the recognizer for $\mathcal{G}$ which is closed under $\circ$.

Proof. To prove (2), let S and T be maximal Ψ-sets of G and H respectively, and let $A = S \cup T$. Then it is easily seen that $G - S$ and $H - T \in \mathrm{acrit}(\Psi)$. Thus $\Psi(G \circ H) \leq |A| + \psi((G \circ H) - A) \leq |S| + |T| + \psi(G - S) + \psi(H - T) = \Psi(G) + \Psi(H)$. The sufficiency of (1) is proved similarly, while that of (3) follows directly from that of (2). QED

When $\circ$ is the union operation, condition (3) is appropriate for κ and α, and condition (1) for I and J. In fact, $m(G \cup H) \leq \max(m(G), m(H))$, and similarly with the parameter $\tilde{m}$. In [13] it was shown that, in general, $I(G \cup H) - m_0 \leq (I(G) - m_0) + (I(H) - m_0)$, and that this is sharp. A similar result holds for $J(G \cup H)$; for sharpness consider $G = K(r-1, r)$ and $H = K(r, r)$. Thus, even with a more restrictive condition on ψ one cannot hope to achieve much more.

Theorem 8 *Let $G \circ H$ denote $G \cup H$ or $G + H$. If $\psi(G \circ H) \geq p(G \circ H) - 1 + \psi_0$ always, then $\Psi(G \circ H) = \min(\Psi(G) + p(H), p(G) + \Psi(H))$ always.*

PROOF. Let A be a Ψ-set of $G \circ H$. Then if $A \not\supseteq V(G)$ and $A \not\supseteq V(H)$, it follows that $|A| + \psi(G \circ H - A) \geq p(G \circ H) - 1 + \psi_0$, which is an upper bound on $\Psi(G \circ H)$. Hence we may assume that $A \supseteq V(G)$ or $A \supseteq V(H)$ and the result follows. QED

Here the join is more relevant, and the theorem applies to all of κ, α, I, J. A consequence of the above result is that for such ψ, $\Psi(G + K_s) = \Psi(G) + s$ and thus $\Psi(K_n) = n - 1 + \psi_0$.

3.2 When $\psi \geq \Delta + \psi_0$

We investigate what happens when $\psi \geq \Delta + \psi_0$. This condition is satisfied by the parameters m and ψ_α, for example, but not by $\tilde{m}$ or ψ_κ.

Clearly, this condition implies that $\Psi \geq \delta + \psi_0$ and that $\Psi' \geq \Delta + \psi_0$. It does not ensure that $\Psi \leq \Psi'$, though. We shall need the following easy lemma.

Lemma 9 *Let $\psi \geq \Delta + \psi_0$, S a Ψ-set of graph G, and $v \in V(G) - S$. Then $\Psi(G) \geq \deg v + \psi_0$.*

The following generalizes a result of [14]:

Theorem 10 *Let ψ satisfy $\psi \geq \Delta + \psi_0$ and let G be a graph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_p$. Then*

$$\Psi(G) \geq \psi_0 + \min_{1 \leq t \leq p} \max(d_t, t - 1).$$

PROOF. Assume that $V(G) = \{v_1, v_2, \ldots, v_p\}$ with $\deg v_i = d_i$ $(1 \leq i \leq p)$, and let S be a Ψ-set of G. Let t be the smallest value such that $v_t \in V(G) - S$. Then by the above lemma, $\Psi(G) \geq \deg v_t + \psi_0$, while clearly $\Psi(G) \geq |S| + \psi_0 \geq t - 1 + \psi_0$. Thus $\Psi(G) \geq \psi_0 + \max(t - 1, d_t)$ and the result follows. QED

This represents an improvement on the bound $\Psi \geq \delta + \psi_0$, and equality is attained for G complete. The next result generalizes one from [13].

Theorem 11 *Let G be a graph and form G' as follows: introduce a new vertex v and join v to r vertices of G. If ψ satisfies $\psi \geq \Delta + \psi_0$ and $r \geq \Psi(G) + 1 - \psi_0$ then $\Psi(G') = \Psi(G) + 1$.*

Proof. Let S be a Ψ-set of G'. If $v \in S$ then it is easily shown that $\Psi(G') = \Psi(G) + 1$. If $v \notin S$, then $\Psi(G') \geq \deg_{G'} v + \psi_0 \geq \Psi(G) + 1$ by Lemma 9, while $\Psi(G') \leq \Psi(G) + 1$ as Ψ is smooth. QED

With the edge analogues, it is easy to see that (in the notation of the theorem) $\Psi'(G') \geq \Psi'(G) + 1$, but there is no guarantee of equality.

3.3 Nordhaus-Gaddum Bounds

We next look at Nordhaus-Gaddum-type lower bounds for Ψ and Ψ'. If ψ satisfies $\psi \geq \Delta + \psi_0$ then $\Psi'(G) + \Psi'(\bar{G}) \geq p(G) - 1 + 2\psi_0$, since necessarily $\Psi' \geq \Delta + \psi_0$. From this one may obtain the Nordhaus-Gaddum lower bounds for edge integrity of [1]. The analogous result holds for Ψ:

Theorem 12 *Let ψ satisfy $\psi \geq \Delta + \psi_0$. Then for all graphs G of order p,*

$$\Psi(G) + \Psi(\bar{G}) \geq p - 1 + 2\psi_0.$$

Proof. Let S and T be Ψ-sets of G and $\bar{G}$ respectively. There are two possibilities. If $S \cup T = V(G)$ then $\Psi(G) + \Psi(\bar{G}) \geq |S| + \psi_0 + |T| + \psi_0 \geq p + 2\psi_0$. On the other hand, if $S \cup T \neq V(G)$ then let $v \in V(G) - (S \cup T)$. Applying Lemma 9, it follows that $\Psi(G) + \Psi(\bar{G}) \geq (\deg_G v + \psi_0) + (\deg_{\bar{G}} v + \psi_0) \geq p - 1 + 2\psi_0$, so that the theorem is proved. QED

This generalizes a result from [12]. Another proof may be obtained from Theorem 10 using straight-forward arithmetic.

We can also establish lower bounds for mean integrity. We shall need the following bound from [7].

Lemma 13 *For all graphs G, $\tilde{m}(G) \geq 1 + 2q(G)/p(G)$.*

Proof. Let G have components with orders $p_1, p_2, \ldots, p_k$. Then $2q(G) \leq 2\sum_i \binom{p_i}{2} = \sum_i p_i^2 - \sum_i p_i = \tilde{m}(G)p(G) - p(G)$. Rearranging yields the bound. QED

Equality holds iff G is the union of cliques.

Theorem 14 *Let G be a graph on p vertices. Then*

$$J(G) + J(\bar{G}) \geq \begin{cases} 3p/4 + 3/2 - 1/(4p) & \text{if } p \text{ is odd,} \\ 3p/4 + 3/2 & \text{if } p \text{ is even,} \end{cases}$$

with equality possible for all p.

Proof. Let S, T be sets of G, $\bar{G}$ respectively for which the mean integrity is attained. Let $A = V - (S \cup T)$ and $D = S \cap T$, and let $|A| = a$, $|D| = d$, $|S| = s$ and $|T| = t$. Then, by the above lemma,

$$\begin{aligned} J(G) + J(\bar{G}) &= s + \tilde{m}(G - S) + t + \tilde{m}(\bar{G} - T) \\ &\geq s + t + 2 + 2(q(G - S) + q(\bar{G} - T))/p. \end{aligned}$$

But, $q(G - S) + q(\bar{G} - T) \geq a(a - 1)/2$. Thus, on letting $x = s + t$, it follows that $J(G) + J(\bar{G})$ is at least as large as the minimum of

$$x + 2 + (p + d - x)(p + d - x - 1)/p,$$

taken over all $0 \leq d \leq x \leq 2p$.

If $x \geq p - 1$ then clearly the smallest possible value is $p + 1$. Consider therefore the case when $x \leq p - 1$. Then the minimum occurs at $d^* = 0$; further minimization yields that $x^* = (p - 1)/2$. If p is odd this is permissible. If p is even, we take the smallest solution for x and d integral. Again $d^* = 0$ while $x^* = p/2$ or $p/2 - 1$. These values give the above lower bounds.

Equality is attainable; if $p = 2l + 1$ then take $G = \bar{K}_l \cup K_{l+1}$, and if $p = 2l$ then $G = \bar{K}_l \cup K_l$. (The J-sets for G are empty; for $\bar{G}$ apply Theorem 8.) QED

In [12] the upper bounds for the parameter integrity were shown to potentially be linked to Ramsey theory. Probably the best bounds for mean integrity would also follow from $J(G) \leq I(G) \leq \alpha(G)$ and (normal) Ramsey bounds.

We define edge mean integrity $J'(G)$ as expected. It has similar Nordhaus-Gaddum bounds to edge integrity:

Theorem 15 *For all graphs G of order p it holds that $J'(G) + J'(\bar{G}) \geq p + 1$, with equality possible for all p.*

Proof. Let S, T be sets of G, $\bar{G}$ respectively for which the edge mean integrity is attained, $|S| = s$ and $|T| = t$. Then by Lemma 13

$$\begin{aligned} J'(G) + J'(\bar{G}) &\geq s + t + 2 + 2(q(G - S) + q(\bar{G} - T))/p \\ &= s + t + 2 + 2(p(p - 1)/2 - s - t)/p \\ &= p + 1 + (s + t)(1 - 2/p). \end{aligned}$$

Thus the lower bound is valid, while equality holds for G or $\bar{G}$ complete only. QED

The upper bound on the sum is $2p$ for p sufficiently large.

4 Further Thoughts

We mention first some examples of Ψ which involve recognizer functions. If $\mathcal{G}$ is the set of graphs which have either at least l components or at most $l-1$ vertices, then the resultant Ψ is the l-connectivity of [6]. Another collection of graphical parameters results from $\mathcal{G}$ being the set of all graphs which do not contain a prescribed F as a subgraph. These Ψ are called F-cover numbers [5]; with $F = K_2$ one obtains the usual vertex cover number.

Other functions ψ might involve the maximum/minimum degrees, the existence of cycles, or the number of cut-vertices. Also, a close relative of integrity and mean integrity is the parameter induced by a "true" average order of a component.

We conclude by considering other generalizations of integrity. A more general class of parameters results if one defines a parameter Φ^f by

$$\Phi^f(G) = \min_{S \subset V(G)} f(|S|, \psi(G-S)),$$

where f is a given function. An example of this is the parameter *toughness* [9]: define $\psi(G)$ to be the number of components of G if G is disconnected, and 0 otherwise, and set $f(x,y) = x/y$. However, it is not immediately evident why one might need this more general definition. Considering real-life situations though, we see that in many networks one may reasonably assume that only a small number of nodes would be incapacitated. Thus one desires a low value of Φ to indicate a "small" S such that $\psi(G-S)$ is small (or high). For instance, the complete bipartite graphs $K(m,m)$ have toughness 1, the same value as the toughness of a (spanning) cycle; this value results from a tough set with cardinality half the order of the graph. Thus in the definition of toughness one might replace $|S|$ with a superlinear $g(|S|)$ such as $|S|^l$ $(l \geq 1)$. One may perform a similar parametrization for integrity, though this lies within the schema already defined.

A different generalization of integrity results from the belief that the parameter m is a good measure of "disconnectedness." The idea is to retain the "m" part of integrity and alter the means by which the values of $|S|$ and $m(G-S)$ are used to obtain a measure of "vulnerability." For example, the vertex cover number is the smallest $|S|$ such that $m(G-S)=1$. We do not pursue these ideas here.

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