Clique/Connected/Total Domination Perfect Graphs

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Abstract. In this note we show that the total domination, connected domination and clique domination numbers equal the domination number in every induced subgraph of a graph iff the graph is both C_5- and P_5-free.

For a graph G, the domination number is denoted by $\gamma(G)$. The total domination number $\gamma_t(G)$ is the minimum cardinality of a dominating set S such that the induced graph $G[S]$ has no isolates. The connected domination number $\gamma_c(G)$ is the same except that $G[S]$ must be connected, the clique domination number $\gamma_{cl}(G)$ requires $G[S]$ to be complete, and the independent domination number $i(G)$ requires $G[S]$ to be edgeless. Clearly $\gamma(G) \le \gamma_t(G) \le \gamma_c(G) \le \gamma_{cl}(G)$ for a connected graph G with $\gamma(G) \ge 2$.

For a parameter π such that $\pi(G) \ge \gamma(G)$ for all G, we say that a graph is π-perfect if $\pi(H) = \gamma(H)$ for all its connected induced subgraphs with $\gamma(H) \ge 2$. In 1979, Sumner and Moore [3] introduced the i-perfect graphs as the *domination perfect* graphs. The problem of characterizing domination perfect graphs turned out to be surprisingly difficult, and a characterization of domination perfect graphs in terms of 17 forbidden subgraphs was eventually obtained by Zverovich and Zverovich [6] in 1995.

In contrast we show that the γ_t-, γ_c- and γ_{cl}-perfect graphs have a simple characterization. The characterization of γ_c-perfect graphs (called *perfect dominant-clique* graphs in [4, 5]) was recently also found by Zverovich [4]. This also implies the characterization of γ_t-graphs.

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Several authors [1, 2, 5] showed that every subgraph of G has a dominating clique if and only if G is P_5- and C_5-free. Cozzens and Kelleher [2] also proved that if G is P_5-, C_5- and Net-free (where the Net is K_3 with feet) then $\gamma(G) = \gamma_{cl}(G)$. Our result shows that the Net is unnecessary.

Theorem. *For a graph G, the following are equivalent:*
(1) *G is total domination perfect;*
(2) *$\gamma(H) = \gamma_t(H)$ for all connected induced subgraphs H with $\gamma(H) = 2$;*
(3) *G is connected domination perfect;*
(4) *G is clique domination perfect;*
(5) *every connected induced subgraph has a dominating clique;*
(6) *G contains no induced P_5 or C_5.*

It is clear that any of the first five conditions implies condition (6). That (6) implies all of the first five is proven below.

Lemma. *Let G be a connected graph with $\gamma(G) \geq 2$. If G has no induced P_5 or C_5, then $\gamma(G) = \gamma_t(G) = \gamma_c(G) = \gamma_{cl}(G)$.*

Proof. While is not hard to prove from scratch, we simplify the proof by using the result of Zverovich [4]. That is, since G has no induced P_5 or C_5, it has a minimum dominating set D such that $G[D]$ is connected.

Suppose D is not a clique. Then there are two nonadjacent vertices of D, say u and v, which have a common neighbor c. But then omitting u implies there is a vertex x adjacent only to u, and similarly a vertex y adjacent only to v. Thus the five vertices $\{u, v, x, y, c\}$ induce a P_5 or C_5, a contradiction. □

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