Note:
Pancyclicity of the Prism

Wayne Goddard [1]

School of Geological and Computer Sciences, University of Natal, Durban, 4041, South Africa

Michael A. Henning [1]

School of Mathematics, Statistics and Information Technology
University of Natal
Private Bag X01
Pietermaritzburg, 3209 South Africa

Abstract

We show that for every 3-connected cubic graph G, the prism $G \times K_2$ has cycles of every even length. Furthermore, if G has a triangle, then $G \times K_2$ is pancyclic.

Key words: Graphs, cubic graphs, prism, cartesian product, pancyclicity

1 Introduction

The prism of a graph G is defined as the cartesian product $G \times K_2$; that is, take two disjoint copies of G and add a matching joining the corresponding vertices in the two copies. Building on earlier work in [1,3,4] and others, Paulraja proved:

Theorem 1 *[5] If G is a 3-connected 3-regular graph, then the prism $G \times K_2$ is hamiltonian.*

In this note we extend Paulraja's result. We show that for every 3-connected 3-regular graph G, the prism $P = G \times K_2$ is "vertex even pancyclic:" for each

[1] Supported in part by the University of Natal and the South African National Research Foundation

vertex v of P there is in P a cycle of every even length from 4 up to the order of P which contains v. Furthermore, if G has a triangle, then P is pancyclic: it has cycles of every length from 3 up to its order.

2 Proofs

In order to establish our results, we need to review the ideas of the proof of Theorem 1. The key idea is to use a spanning cactus of G.

For graph theory terminology we follow [2]. A vertex of degree 1 is called an *end-vertex*. A *block* is a maximal 2-connected subgraph. A *cactus* is a connected graph every block of which is a K_2 or a cycle. A 3-*cactus* is a cactus with maximum degree at most 3 and such that every vertex of degree 3, if any, belongs to a cycle. An *even* 3-*cactus* is a bipartite 3-cactus (that is, every cycle has even length). A *cubic graph* is a graph in which every vertex has degree 3. A spanning subgraph is one with the same vertex set.

Theorem 2 *[5] If G is a 3-connected cubic graph, then there exists a spanning even 3-cactus H.*

Theorem 3 *[5] If H is an even 3-cactus, then the prism $H \times K_2$ is hamiltonian.*

(We note in passing that Fleischner [3] building on work of Goodey & Rosenfeld [4] showed that the prism of a cubic graph is hamiltonian if and only if the graph contains what he called a BEPS-graph. It can be shown that a minimal BEPS-graph is a spanning even 3-cactus, and so a cubic graph is hamiltonian if and only if it contains a spanning even 3-cactus.)

To prove our results, we need the following observation:

Observation 4 *If H is a 3-cactus of order at least 2, then there exists two vertices v_1 and v_2 such that $H - v_1$ and $H - v_2$ are both 3-cacti.*

PROOF. Consider the block graph H' formed by contracting each cycle of H to a single vertex. If H' has one vertex, then H was a cycle, and v_i can be any vertex.

Otherwise, H' contains two end-vertices. Each end-vertex of H' corresponds either to an end-vertex w of H, or to a cycle C in H in which exactly one vertex has degree 3. In the former case, $H - w$ is a 3-cactus; in the latter, $H - x$ is a 3-cactus for any neighbor x of w in C. □

Theorem 5 *If H is an even 3-cactus, then the prism $H \times K_2$ is vertex even pancyclic.*

PROOF. Let v^* be any vertex of $H \times K_2$ and let k be an integer such that $2 \leq k \leq p(H)$. Say v^* corresponds to $v \in V(H)$. By the above observation, H has an even 3-cactus H' of order k containing v. By Theorem 3, $H' \times K_2$ is hamiltonian, and so H has a $2k$-cycle containing v^*. □

As a direct consequence of Theorems 2 and 5 we get:

Theorem 6 *If G is a 3-connected cubic graph, then the prism $G \times K_2$ is vertex even pancyclic.*

Finally, we show that prisms with triangles are pancyclic:

Theorem 7 *If G is a 3-connected cubic graph that contains a triangle, then the prism $G \times K_2$ is pancyclic.*

PROOF. Suppose that G contains a triangle $T = \{x, y, z\}$. It is clear that $T \times K_2$ contains a 5-cycle and that $K_4 \times K_2$ has a 7-cycle. So assume $p(G) \geq 6$ and we wish to find a $(2k+1)$-cycle for $3 \leq k < p(G)$.

Let G' be the graph obtained from G by contracting T to a single vertex t. Assume the two vertices corresponding to t in the prism $G' \times K_2$ are t_1 and t_2. Since G is 3-connected and $p(G) \geq 6$, the graph G' is 3-connected and cubic. By Theorem 2 and the proof of Theorem 5, G' contains a cycle C' of length $2k-2$ that contains both t_1 and t_2.

Now we will convert C' to a cycle of G by replacing $\{t_1, t_2\}$ with five vertices of $T \times K_2$. If C' does not use the edge t_1t_2, then we can replace t_1 by two vertices and t_2 by three vertices and hence this case is handled.

So assume C' uses the edge t_1t_2. Suppose the third neighbours of x, y and z are a, b and c respectively. For each vertex v in G let v_1 and v_2 be the two vertices corresponding to v in $G \times K_2$. Then, without loss of generality we may assume C' contains either the segment a_1, t_1, t_2, a_2 or the segment a_1, t_1, t_2, b_2. These can be replaced by segments $a_1, x_1, y_1, z_1, z_2, x_2, a_2$ and $a_1, x_1, y_1, z_1, z_2, y_2, b_2$ respectively. This completes the proof. □

It is also possible to show that in general if a cubic graph with a triangle has a spanning even 3-cactus then the resultant prism is pancyclic. For example, Fleischner [3] showed that a 2-connected planar cubic graph has a BEPS-graph, and therefore is hamiltonian; so, if a 2-connected planar cubic graph has a triangle then its prism is pancyclic. The situation for nonplanar graphs is unresolved.

Acknowledgements

We would like to thank Moshe Rosenfeld for introducing us to the problem and for helpful discussions. We would also like to thank the referees for several useful comments and for simplifying the proofs.

References

[1] B. Alspach and M. Rosenfeld, On hamiltonian prisms over simple 3-polytopes. *Graphs Combin.* **2** (1986), 1–8.

[2] G. Chartrand and L. Lesniak, *Graphs & Digraphs: Third Edition*, Chapman & Hall, London, 1996.

[3] H. Fleischner, The prism of a 2-connected, planar, cubic graph is hamiltonian (a proof independent of the four colour theorem). In: *Ann. Discrete Math.* **41** (1989), 141–170.

[4] P.R. Goodey and M. Rosenfeld, Hamiltonian circuits in prisms over certain simple 3-polytopes, *Discrete Math.* **21** (1978), 229–235.

[5] P. Paulraja, A characterization of hamiltonian prisms. *J. Graph Theory* **17** (1993), 161–171.