Minimal and Maximal e=1 Functions

P. Dankelmann[0]
University of Natal, Durban
D.J. Erwin[0]
Western Michigan University
G. Fricke
Morehead State University
W. Goddard
University of Natal, Durban
H.C. Swart[0]
University of Natal, Durban

Dedicated to Ernie Cockayne, friend and colleague

ABSTRACT. An $e = 1$ function is a function $f : V(G) \mapsto [0,1]$ such that every non-isolated vertex u is adjacent to some vertex v such that $f(u) + f(v) = 1$, and every isolated vertex w has $f(w) = 1$. A theory of $e = 1$ functions is developed focussing on minimal and maximal $e = 1$ functions. Relationships are traced between $e = 1$ parameters and some well-known domination parameters, which lead to results about classical and fractional domination parameters.

1 Introduction

Let $G = (V, E)$ be a graph. We consider here functions $f : V(G) \mapsto [0,1]$ which we call **fractional set functions** (or, more properly, fractional set functions of G). Such functions arise in fractional domination, fractional packing and many other parameters. For a good survey of fractional graph theory see the book by Scheinerman and Ullman [SU97].

In particular, we introduce fractional set functions with the property that every non-isolated vertex v has a neighbour w whose **weight** $f(w)$ is $1 - f(v)$. We call such a function an $e = 1$ **function**. More precisely, f is an $e = 1$ function of graph G if

$$\forall v \in V(G) \begin{cases} \deg v = 0 & \Longrightarrow \ f(v) = 1, \\ \deg v > 0 & \Longrightarrow \ \exists u \in N(v) \text{ such that } f(u) + f(v) = 1. \end{cases}$$

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For example, if G has no isolated vertices then the all-1/2 function is an $e = 1$ function.

If f and g are two fractional set functions of G, then we say that $f \leq g$ if $f(v) \leq g(v)$ for all $v \in V(G)$. We say that $f < g$ if $f \leq g$ and $f(u) < g(u)$ for some $u \in V(G)$. Then, an $e = 1$ function f is a **maximal e=1 function** if there exists no $e = 1$ function g such that $g > f$. An $e = 1$ function f is a **minimal e=1 function** if there exists no $e = 1$ function g such that $g < f$.

These functions are connected with domination as follows. A fractional set function is called a **fractional dominating function** if for every $v \in V(G)$ it holds that $f(N[v]) \geq 1$. First, the characteristic function of a minimal dominating set of G is a minimal $e = 1$ function of G. On the other hand, an $e = 1$ function is also a fractional dominating function.

There is also a connection with independence and vertex cover. A fractional set function is called a **fractional vertex covering function** if for every edge $uv \in E(G)$ it holds that $f(u) + f(v) \geq 1$. For a graph without isolated vertices, every minimal fractional vertex covering function is an $e = 1$ function. At the same time, the characteristic function of a maximal independent set of G is a minimal $e = 1$ function of G.

We will denote the domination number of G by $\gamma(G)$, the fractional domination number by $\gamma_f(G)$, the upper domination number (maximum cardinality of a minimal dominating set) by $\Gamma(G)$, and the upper fractional domination number (maximum weight of a minimal fractional dominating function) by $\Gamma_f(G)$. We will denote the vertex cover number of G by $\alpha(G)$.

The paper is organised as follows. In Section 2 we define $\gamma_{\underline{e=1}}(G)$ and $\Gamma_{\overline{e=1}}(G)$ as the minimum and maximum values of an $e = 1$ function, and show inter alia that $\gamma_{\underline{e=1}}(G) = \gamma(G)$ and that for graphs without isolated vertices $\gamma_{\underline{e=1}}(G) + \Gamma_{\overline{e=1}}(G) = p(G)$. In Section 3 we define $\gamma_{\overline{e=1}}(G)$ and $\Gamma_{\underline{e=1}}(G)$ as the infimum weight of a maximal $e = 1$ function and the supremum weight of a minimal $e = 1$ function respectively, and show that $\gamma_{\overline{e=1}}(G) + \Gamma_{\underline{e=1}}(G) = p(G)$ for a graph without isolated vertices. In Section 4 we further explore the value of $\gamma_{\overline{e=1}}(G)$ and show that the infimum in the definition cannot be replaced by a minimum. Thereafter we prove bounds on $\gamma_{\overline{e=1}}(G)$, in particular that $\gamma_{\overline{e=1}}$ for a path is approximately 3/7 its order. Section 5 suggests some further questions.

2 Minimum and Maximum e=1 Functions

If $S \subseteq V(G)$ is a set of vertices, we denote by $f(S)$ the sum $\sum_{v \in S} f(v)$. We define the **total weight** of f as $f(V(G))$ and denote it by $|f|$.

Then we define

$$\gamma_{\underline{e=1}}(G) = \inf\{\,|f| : f \text{ an } e = 1 \text{ function}\,\},$$
$$\Gamma_{\overline{e=1}}(G) = \sup\{\,|f| : f \text{ an } e = 1 \text{ function}\,\},$$

Recall that the function which is $1/2$ on non-isolates and 1 on isolated vertices in an $e = 1$ function. It follows that:

$$\gamma_{\underline{e=1}}(G) \leq \frac{p(G) + \mathrm{iso}(G)}{2} \leq \Gamma_{\overline{e=1}}(G), \tag{2.1}$$

where $p(G)$ denotes the order of graph G and $\mathrm{iso}(G)$ the number of isolated vertices.

Consider as a simple example the complete graph K_n on n vertices. It is not hard to see that the total weight of an $e = 1$ function must be at least 1 for any graph, and this is achieved for the complete graph by assigning the weight 1 to one vertex and weight 0 to the remainder. Similarly, the weight of an $e = 1$ function is at most $n - 1$ if the graph is nonempty, and this is acheived by the complete graph by assigning the weight 0 to one vertex and weight 1 to the remainder. Thus

$$\gamma_{\underline{e=1}}(K_n) = 1 \qquad \text{and} \qquad \Gamma_{\overline{e=1}}(K_n) = n - 1$$

for $n \geq 2$.

Our first result shows that the minimum weight of an $e = 1$ function is equal to the domination number of the graph.

Theorem 2.1 *For any graph G,*

$$\gamma_{\underline{e=1}}(G) = \gamma(G).$$

PROOF. Since $\gamma_{\underline{e=1}}(K_1) = 1 = \gamma(K_1)$, it suffices to prove the statement for graphs G with no isolated vertices.

Now, the characteristic function of a minimal dominating set of G is an $e = 1$ function of G. So

$$\gamma_{\underline{e=1}}(G) \leq \gamma(G).$$

It remains to prove that $\gamma_{\underline{e=1}}(G) \geq \gamma(G)$ for a graph without isolates.

Let real number $\varepsilon > 0$ be given. By the definition of $\gamma_{\underline{e=1}}(G)$, there is an $e = 1$ function f with $|f| < \gamma_{\underline{e=1}}(G) + \varepsilon$. Let $a_1, a_2, \ldots, a_k$ denote the values of f which are at most $1/2$. Denote by G_i the subgraph induced by all edges joining a vertex of weight a_i to a vertex of weight $1 - a_i$. If

$a_i < 1/2$ then G_i is bipartite and has a bipartition (V_i, W_i) with $|V_i| \geq |W_i|$. If $a_i = 1/2$ then let F_i be a spanning forest of G_i: this has a bipartition (V_i, W_i) with $|V_i| \geq |W_i|$.

We define a fractional set function g by assigning the weight 0 to each vertex in $\bigcup_{i=1}^{k} V_i$ and 1 to each vertex in $\bigcup_{i=1}^{k} W_i$. It follows that $|g| \leq |f|$, since

$$f(V(G_i)) \geq |W_i| + (|V_i| - |W_i|)a_i \geq |W_i| = g(V(G_i)).$$

Next we note that $\bigcup_{i=1}^{k} W_i$ is a dominating set of G, since every W_i is a dominating set in G_i and $\bigcup_{i=1}^{k} G_i$ is a spanning subgraph of G.

Hence we have

$$\gamma(G) \leq |\bigcup_{i=1}^{k} W_i| = |g|,$$

and hence $\gamma(G) \leq |f| < \gamma_{\underline{e=1}}(G) + \varepsilon$. Since this is true for all $\varepsilon > 0$, it follows that $\gamma(G) \leq \gamma_{\underline{e=1}}(G)$.

Inequality 2.1 and the above theorem yield the following result originally due to Ore [Ore62].

Corollary 2.2 *For a graph G without isolated vertices, $\gamma(G) \leq p(G)/2$.*

For the complete graph it held that $\gamma_{\underline{e=1}}(K_n) + \Gamma_{\overline{e=1}}(K_n) = n$. This pattern is no coincidence. For: given a fractional set function f, we may define the **complementary** fractional set function f_{not} by

$$f_{not}(v) = \begin{cases} 1 & \text{if } \deg v = 0 \\ 1 - f(v) & \text{if } \deg v > 0. \end{cases}$$

It follows immediately that f is an $e = 1$ function if and only if f_{not} is an $e = 1$ function. Furthermore, f is a minimal (maximal) $e = 1$ function if and only if f_{not} is a maximal (minimal) $e = 1$ function.

It follows that:

Theorem 2.3 *For any graph G, $\Gamma_{\overline{e=1}}(G) + \gamma_{\underline{e=1}}(G) = p(G) + \mathrm{iso}(G)$.*

That is:

Corollary 2.4 *For any graph G, $\Gamma_{\overline{e=1}}(G) = p(G) - \gamma(G) + \mathrm{iso}(G)$.*

Now there is a further connection with domination:

Theorem 2.5 *For any graph G, $\Gamma_{\overline{e=1}}(G) \geq \Gamma_f(G)$.*

PROOF. It sufficies to prove the theorem for connected G. Let f be a minimal fractional dominating function of G. Let $E' = \{ uv \in E(G) \mid f(u) + f(v) \leq 1 \}$. We show that every $v \in V(G)$ is incident with some $e \in E'$. If $f(v) = 0$, then every edge incident with v is in E'. If $f(v) > 0$, then by minimality either $f(N[v]) = 1$, so that every edge incident with v is in E', or v has a neighbour w with $f(N[w]) = 1$, so that $vw \in E'$. That is, E' covers $V(G)$.

Let E'' be a minimal subset of E' that covers $V(G)$. Then the graph $\langle E'' \rangle$ induced by E'' is a disjoint union of stars. For every component of $\langle E'' \rangle$ with central vertex w, we define $g(w) = f(w)$ and $g(w_i) = 1 - f(w)$ for the neighbours w_i of w. It follows that g is an $e = 1$ function of G and that $g(v) \geq f(v)$ for all $v \in V(G)$. Hence $\Gamma_{\overline{e=1}}(G) \geq |f|$.

By combining the above two results, we obtain the following result, which can also be deduced by combining bounds on the irredundance number given in [CFPT81, DHLF91]. (We thank Michael Henning for pointing this out.)

Corollary 2.6 *For a graph G without isolates $\gamma(G) + \Gamma_f(G) \leq p(G)$.*

We now have the following chain of inequalities:

$$\gamma_f(G) \leq \gamma_{\underline{e=1}}(G) = \gamma(G) \leq \Gamma(G) \leq \Gamma_f(G) \leq \Gamma_{\overline{e=1}}(G) = p(G) - \gamma(G) + \text{iso}(G). \tag{2.2}$$

As a final result in this section, we observe a connection between $e = 1$ functions and fractional packing functions. A fractional set function is a fractional **packing** function if for every $v \in V(G)$ it holds that $f(N[v]) \leq 1$.

Theorem 2.7 *Let G be a connected graph that is not K_2. Then f is both an $e = 1$ function and a fractional packing function if and only if f is the characteristic function of an efficient dominating set.*

PROOF. It is immediate that the characteristic function of an efficient dominating set is both an $e = 1$ function and a fractional packing function.

So let f be both an $e = 1$ function and a fractional packing function. Let $v \in V(G)$ be any vertex. Since f is an $e = 1$ function, v has a neighbour w such that $f(v) + f(w) = 1$. Since f is a fractional packing function, all other neighbours of v or w have weight 0. In particular, any vertex has at most one neighbour of positive weight.

Since G is not K_2, there is another vertex x adjacent to one of v or w. We know $f(x) = 0$. Since x has only one neighbour of positive weight, all neighbours of x have weight 0 or 1, including v. In particular, any vertex has weight either 0 or 1.

So f is the characteristic function of a set which is both a dominating set and a packing; that is, an efficient dominating set.

3 Minimal and Maximal $e = 1$ functions

We define

$$\gamma_{\overline{e=1}}(G) = \inf\{\, |f| : f \text{ a maximal } e = 1 \text{ function} \,\},$$
$$\Gamma_{\underline{e=1}}(G) = \sup\{\, |f| : f \text{ a minimal } e = 1 \text{ function} \,\}.$$

Since the function which is 1/2 on non-isolates and 1 on isolated vertices is both a minimal and a maximal $e = 1$ function, it follows that:

$$\gamma_{\underline{e=1}}(G) \le \gamma_{\overline{e=1}}(G) \le \frac{p(G) + \mathrm{iso}(G)}{2} \le \Gamma_{\underline{e=1}}(G) \le \Gamma_{\overline{e=1}}(G). \qquad (3.1)$$

Furthermore, since f is a minimal (maximal) $e = 1$ function if and only if f_{not} is a maximal (minimal) $e = 1$ function, and $|f| + |f_{not}| = p(G) + \mathrm{iso}(G)$, it follows that:

$$\Gamma_{\underline{e=1}}(G) = p(G) - \gamma_{\overline{e=1}}(G) + \mathrm{iso}(G).$$

We will focus our attention on $\gamma_{\overline{e=1}}(G)$; results about $\Gamma_{\underline{e=1}}(G)$ can be deduced from the above formula.

Consider some examples. We first consider the complete graph on n vertices. It is not hard to see that in a maximal $e = 1$ function that is not the all-1/2 function, one vertex, v say, has weight less than 1/2, and the remaining vertices have weight $1 - f(v)$. The total weight is minimised when $f(v) = 1/2$ and so $\gamma_{\overline{e=1}}(K_n) = n/2$.

For a more interesting example, consider the graph H formed by taking the disjoint union of a complete graph on four vertices and a path on three vertices and identifying the central vertex of the path and one vertex of the clique. See Figure 1. Some work shows that:

$$\gamma_{\overline{e=1}}(H) = 5/2,$$

which is achieved by the weight 1 on the central vertex, 0 on the two end-vertices, and 1/2 on the remaining three vertices.

We observed earlier that in a graph without isolated vertices, the characteristic function of a minimal vertex cover is an $e = 1$ function of G, in fact a maximal one. It follows that:

Theorem 3.1 *For any graph G, $\gamma_{\overline{e=1}}(G) \le \alpha(G) + \mathrm{iso}(G)$.*

Now, the characteristic function of a minimal dominating set of G is a minimal $e = 1$ function of G. It follows that:

$$\Gamma_{\underline{e=1}}(G) \ge \Gamma(G).$$

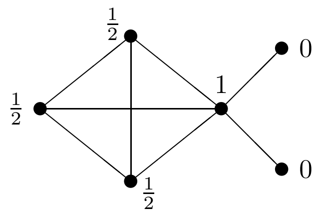

Figure 1: A graph

4 The Value of Minimum Maximal $e = 1$ Functions

Assume throughout that G is a graph without isolated vertices.

Consider an $e = 1$ function f. It is called **pure** if it uses only weights in the codomain $\{0, \frac{1}{2}, 1\}$. We say a vertex x **depends** on a vertex y if y is the only neighbour of x with weight $1 - f(x)$. We let S_f, M_f and L_f (small, medium and large) be the set of vertices of weight less than 1/2, exactly 1/2, and more than 1/2 respectively. We will omit the subscript if f is clear from context.

The following theorem gives an upper bound on the minimum value of a maximal $e = 1$ function. Note that the empty set is considered to be independent.

Theorem 4.1 *Let G be a graph without isolated vertices. If $P(G)$ denotes the minimum value of a pure maximal $e = 1$ function of G, then*

$$\gamma_{\overline{e=1}}(G) \leq P(G) = \frac{1}{2}\left(\min_{T \textit{ indep}} p(G) + |N(T)| - |T|\right).$$

PROOF. Clearly, $\gamma_{\overline{e=1}}(G) \leq P(G)$.

Consider any pure maximal $e = 1$ function f. Let S_D denote the set of vertices with weight 0 which have a vertex depending on them. For each vertex $u \in S_D$, let l_u be one vertex which depends on it, and let L_D be the set $\{\, l_u : u \in S_D \,\}$. Note that the vertices l_u are distinct and thus $|S_D| = |L_D|$.

Let S_I denote the set of the remaining vertices with weight 0. If $v \in S_I$, then since no vertex depends on v, by maximality all neighbours of v have weight 1. In particular, S_I is an independent set. Furthermore, v cannot have a neighbour in L_D; so $N(S_I) \subseteq L - L_D$.

By the above discussion, the average weight of a vertex in each of the three sets M, $S_D \cup L_D$ and $L - L_D - N(S_I)$ is at least 1/2. Thus the average weight outside $S_I \cup N(S_I)$ is at least 1/2. So the total weight of f is at least

$$|f| \geq \frac{1}{2}(p(G) + |N(S_I)| - |S_I|).$$

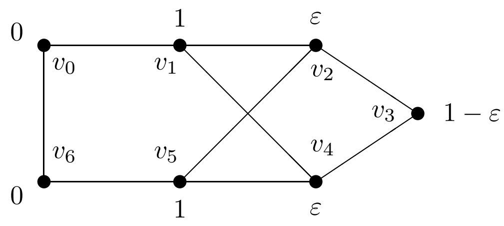

Figure 2: A graph H with no minimum maximal $e = 1$ function

Since this holds for all pure maximal $e = 1$ functions f, it follows that $P(G) \geq \text{RHS}$ (with $T = S_I$).

On the other hand, let T be any independent set achieving the RHS bound. Clearly T is maximal with respect to the given neighbourhood. Define a fractional set function g by assigning the vertices of T weight 0, the vertices of $N(T)$ weight 1, and the remaining vertices weight 1/2. This is an $e = 1$ function: every vertex of weight 0 has only neighbours of weight 1, every vertex of weight 1 is adjacent to a vertex of weight 0 by definition, and every vertex of weight 1/2 has another neighbour of weight 1/2 (else it could be added to T without increasing $|N(T)|$).

Furthermore, the function g is maximal: a vertex of weight 0 cannot have its weight increased, because all neighbours already have weight 1, and a weight 1/2 cannot be increased because all neighbours already have weight at least 1/2. This shows that $P(G) \leq \text{RHS}$.

It can be shown that: if the graph is bipartite, the optimal T is a maximum independent set. So, in this case P equals the vertex cover number, and the bound of Theorem 3.1 is recovered.

Further, it can be shown that if there is a maximal $e = 1$ function of weight exactly $\gamma_{\overline{e=1}}(G)$, then there is a pure $e = 1$ function of weight $\gamma_{\overline{e=1}}(G)$ so that $\gamma_{\overline{e=1}}(G) = P(G)$.

Surprisingly perhaps, the value of $\gamma_{\overline{e=1}}(G)$ is not always attained and the infimum is a true infimum. Consider for example, the graph H of Figure 2. This can be thought of as the 7-cycle $v_0, v_1, \ldots, v_6, v_0$ with the chords v_1v_4 and v_2v_5. There, $\gamma_{\overline{e=1}}(H) = 3$ which is reached as $\varepsilon \to 0$ in the indicated weighting. (That 3 is a lower bound follows from the fact that $\gamma_{\underline{e=1}}(H) = \gamma(H) = 3$.) On the other hand, $P(H) = p(H)/2 = 7/2$, as no independent set is larger than its neighbourhood.

We show next that in general a near-optimal maximal $e = 1$ function can be chosen to be nearly pure.

Proposition 4.2 *For any $\varepsilon > 0$, there is a maximal $e = 1$ function f*

with total weight at most $\gamma_{\overline{e=1}}(G)+\varepsilon$ *and which takes values only in the codomain* $[0,\varepsilon)\cup(1/2-\varepsilon,1/2+\varepsilon)\cup(1-\varepsilon,1]$.

Furthermore, $\gamma_{\overline{e=1}}(G)$ *is half-integral, and if* ε *is sufficiently small, then* $\gamma_{\overline{e=1}}(G)=|L_f|+|M_f|/2$.

PROOF. Let real number $\varepsilon>0$ be given. By the defintion of $\gamma_{\overline{e=1}}(G)$ there is a maximal $e=1$ function f with total weight at most $\gamma_{\overline{e=1}}(G)+\varepsilon$. Let S'_f denote the set of vertices with weight in $[\varepsilon,1/2-\varepsilon]$, and L'_f denote the set of vertices with weight in $[1/2+\varepsilon,1-\varepsilon]$. Let f be such an $e=1$ function such that S'_f is as small as possible. Suppose S'_f not empty.

Let a' denote the maximum weight of a vertex with weight less than ε (if no such vertex then let $a'=0$), let a denote the minimum weight of a vertex with weight at least ε, let b denote the maximum weight of a vertex with weight at most $1/2-\varepsilon$, and let b' denote the the minimum weight of a vertex with weight more than $1/2-\varepsilon$ but at most $1/2$ (if no such vertex then let $b'=1/2$). Note that $0\le a'<\varepsilon\le a\le b\le 1/2-\varepsilon<b'\le 1/2$.

For real number δ, we define a set function g_δ by:

$$g_\delta(v)=\begin{cases} f(v)+\delta & \text{if } f(v)\in L'_f\\ f(v)-\delta & \text{if } f(v)\in S'_f\\ f(v) & \text{otherwise.}\end{cases}$$

Assume $b-b'<\delta<a-a'$. Then, it is easily checked that the range of g_δ is contained in $[0,1]$, and that g_δ is an $e=1$ function. Also, the numerical ranking of the vertices under g_δ is the same as under f, and so g_δ is a maximal $e=1$ function. Furthermore, the weight of g_δ is

$$|f|+(|L'_f|-|S'_f|)\delta.$$

So, if $|L'_f|\le|S'_f|$, then g_δ contradicts the choice of f for δ sufficiently close to $a-a'$. On the other hand, if $|L'_f|\ge|S'_f|$, then g_δ contradicts the choice of f for δ sufficiently close to $b-b'$. So, in fact, S'_f must be empty and f is the $e=1$ function we desire.

Now, since we can force each weight arbitrarily close to a half-integral value, $\gamma_{\overline{e=1}}(G)$ is half-integral. (Suppose $\gamma_{\overline{e=1}}(G)$ is not half-integral and let ε be the gap between it and the nearest half-integer. Then we can by above find a maximal $e=1$ function, with total weight in the range $[\gamma_{\overline{e=1}}(G),\gamma_{\overline{e=1}}(G)+\varepsilon/2)$, and with all weights within $\varepsilon/(100p)$ of a half-integer. But the latter says the total weight is within $\varepsilon/100$ of a half-integer, a contradiction.)

If ε is sufficiently small, then $|f|$ is very close to $|L_f|+|M_f|/2$, and since $\gamma_{\overline{e=1}}(G)$ is half-integral and very close to $|f|$, it follows that $\gamma_{\overline{e=1}}(G)=|L_f|+|M_f|/2$, as required.

As another example of the behaviour of near-optimal maximal $e = 1$ functions, consider the following. Recall the graph H of Figure 2. Form graph H' by adding to H a clique on m vertices and adding one edge from the clique (say from vertex x) to v_0 (which receives weight 0 in H). A simple but time-consuming check shows that if m is sufficiently large then $\gamma_{\overline{e=1}}(H') = (p(H') - 1)/2$, and in a near-optimal maximal $e = 1$ function f the weight of x is $1/2 - \delta$ and the weight of every other vertex in the clique is $1/2 + \delta$ for some δ between 0 and ε.

A lower bound follows from the above proposition:

Proposition 4.3 *Let $X(G)$ denote the minimum value of $|L| + |M|/2$ over all colourings of $V(G)$ with three colour classes S, L and M such that:*
if S_D is the subset of S of vertices with a non-L neighbour, then there is a subset L_D of L such that (the graph induced by the edges) $[S_D, L_D]$ has a matching and every vertex of $S - S_D$ has a neighbour in $L - L_D$.

Then $\gamma_{\overline{e=1}}(G) \geq X(G)$.

Proof. Consider a near-optimal maximal $e = 1$ function f as given by the above proposition. The partition of $V(G)$ into S_f, M_f and L_f is a three-colouring. So consider a vertex u of S_D. Since it has a non-L-neighbour and its value cannot be increased, there is a vertex dependent on it. Such a vertex must be in L; choose one and call it l_u. Let L_D be the set of all vertices l_u for $u \in S_D$.

Furthermore, consider a vertex v of $S - S_D$. It is adjacent to a vertex l_v of weight $1 - f(v)$. This vertex cannot be in L_D, by the definition of dependency. Thus the colouring induced by f has the desired properties. Hence, $X(G) \leq |L_f| + |M_f|/2 = \gamma_{\overline{e=1}}(G)$, as required.

For the graph H in Figure 2, $X(H) = 3$.

Corollary 4.4 *(a) For the complete graph $\gamma_{\overline{e=1}}(K_n) = n/2$ for $n \geq 2$.*
(b) For the complete bipartite graph $\gamma_{\overline{e=1}}(K(a, b)) = \min\{a, b\}$.

Proof. (a) By the chain 3.1 of inequalities, $\gamma_{\overline{e=1}}(K_n) \leq n/2$ for $n \geq 2$. The lower bound follows from the above proposition since if $|S| > 1$ then $S = S_D$ and $|L| \geq |S|$.

(b) By Theorem 3.1 $\gamma_{\overline{e=1}}(K(a, b)) \leq \min\{a, b\}$. On the other hand, by the above proposition a maximal $e = 1$ function has value less than half the order only if $S - S_D$ is nonempty; this means that $S \cup M$ is confined to one partite set and L is at least all the other partite set.

Proposition 4.5 *For the path on n vertices, $\gamma_{\overline{e=1}}(P_n) = 3n/7 \pm O(1)$.*

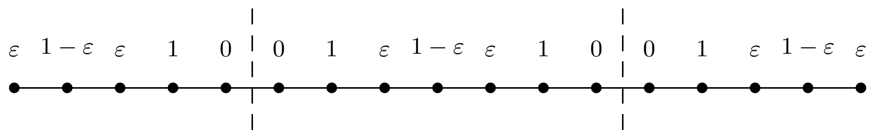

Figure 3: A maximal $e = 1$ function on a path

PROOF. Consider the colouring from Proposition 4.3. We show that $X(G) \geq 3n/7 - O(1)$. Partition the vertices of the path into subpaths as follows: remove each edge joining two vertices in the same colour and each edge incident with an M-vertex. Let P be a subpath with average weight less than $1/2$. Then in P the vertices of S and L alternate, starting and ending with S. Say P is $v_1 v_2 \ldots v_{2i+1}$.

Assume P is neither the first nor the last subpath. Then v_1, which is in S, has a neighbour outside the subpath; since this neighbour is not in L it follows that $v_1 \in S_D$. This means that $i \geq 1$ and vertex $v_2 \in L_D$ (as the match for v_1). The next vertex v_3 cannot be in S_D (since then it would require an L-neighbour other than v_2). This means that $i \geq 2$, and that $v_4 \notin L_D$. Similarly, the last vertex $v_{2i+1} \in S_D$ while its neighbour $v_{2i} \in L_D$. It follows that $i \geq 3$. That is, for each subpath other than the first and last, the average weight of the subpath is at least $3/7$.

If P is the first or last subpath but is not the only subpath, then it follows by similar reasoning that $i \geq 2$ and the average weight of P is at least $2/5$. But this is only attainable for a length-5 subpath. If P is the unique subpath then the average weight is at least $3/7$ provided $n \geq 7$. Hence we have shown that the average weight of each subpath is at least $3/7$ except possibly for subpaths with a total of 10 vertices. The lower bound on $X(G)$ follows.

If $n \geq 10$ and $n \equiv 3 \bmod 7$, then there is a maximal $e = 1$ function with weight $\lfloor 3n/7 \rfloor$ as follows: the first subpath is $\varepsilon, 1-\varepsilon, \varepsilon, 1, 0$; other subpaths are $0, 1, \varepsilon, 1-\varepsilon, \varepsilon, 1, 0$; and the last subpath $0, 1, \varepsilon, 1-\varepsilon, \varepsilon$. See Figure 3 for the weighting which shows that $\gamma_{\overline{e=1}}(P_{17}) = 7$. For other values of n, a similar weighting shows that $\gamma_{\overline{e=1}}(P_n) \leq 3n/7 + O(1)$.

It seems unlikely that there is always equality in Proposition 4.3.

5 Possible Future Work

We suggest a few ideas which may bear investigation.

In [Sla96, GH99] and elsewhere, the closed unit interval we have studied is replaced by an arbitrary subset of the real numbers. This technique could be used to generate "real" parameters corresponding to the fractional parameters discussed here.

Also, we do not know whether the function $\min_{\mathrm{T\ indep}} |N(T)| - |T|$ has been studied. The maximum value was studied by Slater [Sla77].

Also, there is the question of the complexity of calculating $\gamma_{\overline{e=1}}(G)$, probably NP-complete.

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