

Note: Simultaneous Graph Parameters: Factor Domination and Factor Total Domination

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Abstract

Let F_1, F_2, \dots, F_k be graphs with the same vertex set V . A subset $S \subseteq V$ is a factor dominating set if in every F_i every vertex not in S is adjacent to a vertex in S , and a factor total dominating set if in every F_i every vertex in V is adjacent to a vertex in S . The cardinality of a smallest such set is the factor (total) domination number. In this note we investigate bounds on the factor (total) domination number. These bounds exploit results on colorings of graphs and transversals of hypergraphs.

Key words: cycles, bounds, factors, total domination

1 Introduction

Given a collection of graphs F_1, \dots, F_k on the same vertex set V , one can consider those subsets of V that have a desired property simultaneously in all graphs. For example, one might want a set which is independent in all graphs simultaneously. But this is simply an independent set in the edge-union of the graphs.

A more interesting example is to consider a set which dominates all the graphs simultaneously. This was first explored by Brigham and Dutton [3] (who termed this a factor dominating set) and Sampathkumar [10] (who termed it a global dominating set). The natural question is what is the minimum size

of a simultaneous dominating set: following [3] we call this the factor domination number. This question has been studied in [2] and [8, Section 7.6]. Most recently some bounds were established in [7].

In this paper we extend the results on factor domination and investigate the equivalent notion for total dominating sets. In particular, we investigate bounds in terms of the minimum degrees and order of the factors.

1.1 Definitions and Notation

We generally use the definitions and terminology of [8]. In particular, if $G = (V, E)$ denotes a graph, then the (open) neighborhood of vertex $v \in V$ is denoted by $N(v)$ while $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The set S is a *dominating set* (DS) if $N[S] = V$, and a *total dominating set* (TDS) if $N(S) = V$. The *domination number* $\gamma(G)$ and *total domination number* $\gamma_t(G)$ are the minimum cardinality of a DS and TDS respectively. For a survey see [8,9].

A *factoring* is a collection F_1, F_2, \dots, F_k of (not necessarily edge-disjoint) graphs with common vertex set V (the union of whose edge sets is not necessarily the complete graph). The *combined graph* of the factoring has vertex set V and edge set $\bigcup_{i=1}^k E(F_i)$. Each F_i is a *factor*.

A subset $S \subseteq V$ is a *factor dominating set* (FDS) of the factoring if S is simultaneously a DS of F_i for all $1 \leq i \leq k$. The cardinality of a smallest such set, denoted by $\gamma(F_1, F_2, \dots, F_k)$, is the *factor domination number* of the factoring. Similarly, one can define a *factor total dominating set* (FTDS) (provided each F_i is isolate-free), and the *factor total domination number*. Directly from the definition we obtain

$$\max_{1 \leq i \leq k} \gamma_t(F_i) \leq \gamma_t(F_1, F_2, \dots, F_k) \leq \sum_{i=1}^k \gamma_t(F_i).$$

Let $H = (X, C)$ be a hypergraph with vertex set X and hyperedge (multi-)set C . A set $S \subseteq X$ is a *transversal* of H if S contains at least one vertex from every hyperedge of H . The *transversal number* $\tau(H)$ is the minimum cardinality of a transversal of H . If H is a graph then this is just the vertex cover number, denoted $\alpha(H)$.

Given a graph $G = (V, E)$, the *open neighborhood hypergraph* (ONH) is the hypergraph $H = (V, C)$ where C is the (multi-)set of open neighborhoods $N(v)$ of vertices $v \in V$. This hypergraph has n vertices and n hyperedges. A total dominating set of G is clearly equivalent to a transversal of H and

$\gamma_t(G) = \tau(H)$. This fact is exploited for example in [13]. In the case where G is 2-regular, then H is a graph (sometimes called the proper square of G): two vertices of H are adjacent if and only if they have a common neighbor in G .

The minimum and maximum degrees of the graph G are $\delta(G)$ and $\Delta(G)$, respectively. The graph is *d-degenerate* if every subgraph has minimum degree at most d . The *independence number* $\beta(G)$ of G is the maximum cardinality of an independent set of vertices of G . An *end-vertex* is a vertex of degree one.

2 Factor Domination Revisited

In [7] the following bound was established:

Theorem 1 ([7]) *Let F_1, F_2 be isolate-free factors of K_n . Then $\gamma(F_1, F_2) \leq 2n/3$, and this is sharp.*

The authors went on to provide upper bounds for the factor domination number of k isolate-free factors of the form $(1 - O(3^{-k/2}))n$. We provide an improvement. For this purpose, we define a *star-forest* as a graph whose every component is a star.

Theorem 2 *If G is the combined graph of k star-forests, $k \geq 3$, then G is $(2k - 2)$ -colorable.*

PROOF. By induction on the order. Assume G has minimum degree δ . If $\delta \leq 2k - 3$, then we can induct: remove a minimum-degree vertex v , color the graph $G - v$ using $2k - 2$ colors, and add v back in. So we may assume $\delta \geq 2k - 2$.

Define an *NSE (nontrivial star edge)* as an edge in a star with size at least 2. Since $k \geq 3$ we have $\delta > k$. So, every vertex v is the center of a nontrivial star in at least one of the factors, and thus an end-vertex in at most $k - 1$ factors. In particular, there are at most $n(k - 1)$ NSEs. Also, v is the center vertex on at least $\deg(v) - (k - 1)$ NSEs, and so there are at least $\sum_v \deg(v) - n(k - 1)$ NSEs. It follows that $n\delta \leq \sum_v \deg(v) \leq 2n(k - 1)$. Hence G is $(2k - 2)$ -regular; further, every edge is an NSE and every vertex is an end-vertex in exactly $k - 1$ factors.

Now suppose G contains a K_{2k-1} component C . Then each vertex in C is the center of a nontrivial star in exactly one factor. Since $2k - 1 > k$, by the Pigeonhole Principle, two vertices must be the center of stars in the same factor, and thus cannot be adjacent, a contradiction. Hence G does not contain a K_{2k-1} component. By Brooks' theorem [4], G is $(2k - 2)$ -colorable. \square

If $k = 2$, then it is known that G is 3-colorable [11,12].

Theorem 3 *Let F_1, \dots, F_k , $k \geq 3$, be isolate-free factors of K_n . Then $\gamma(F_1, \dots, F_k) \leq (2k - 3)n/(2k - 2)$, and this is sharp.*

PROOF. Since F_i is isolate-free, every vertex cover of F_i is a dominating set. In particular, a vertex cover of the combined (multi-)graph G of F_1, F_2, \dots, F_k is a FDS of the factoring, and so $\gamma(F_1, \dots, F_k) \leq \alpha(G)$.

We may assume that each F_i is a minimal isolate-free graph and thus a star-forest. Hence, by the above result, G is $(2k - 2)$ -colorable. Thus, $\beta(G) \geq n/(2k - 2)$, and so $\alpha(G) \leq n(2k - 3)/(2k - 2)$, as required.

The sharpness follows since we can make the combined graph G a union of K_{2k-2} . A K_{2k-2} can be obtained as follows. Take sets $A = \{a_i\}$ and $B = \{b_i\}$ of $k - 1$ vertices. For $1 \leq i \leq k - 1$, let factor F_i be all edges $a_i a_j$ and $b_i b_j$ for $j < i$ and $a_i b_j$ and $b_i a_j$ for $j > i$. Finally let F_k consist of all the edges $a_i b_i$. \square

3 Total Factor Domination

We provide upper bounds on the factor total domination number in terms of the smallest minimum degree and the order.

3.1 Connectivity

Since a graph G with minimum degree one and order n can have $\gamma_t(G) = n$, there is no better upper bound on the factor total domination number than the order. If G is connected, however, then $\gamma_t(G) \leq 2n/3$ provided $n \geq 3$ (see [6]). Nevertheless, if n is even, it is easy to find two edge-disjoint connected factors $F_1 = (V, E_1)$ and $F_2 = (V, E_2)$ such that $\gamma_t(F_1, F_2) = n$, as follows.

Partition V into two equal sets V_1 and V_2 . Let E_1 consist in all edges joining vertices of V_1 along with a matching between V_1 and V_2 ; let E_2 consist in all edges joining vertices of V_2 along with a matching between V_1 and V_2 . Thus each vertex in V is adjacent to an end-vertex in either F_1 or F_2 . Since any TDS contains all such vertices, it follows that any FTDS of the factoring $\{F_1, F_2\}$ must contain the entire set V , as required. This yields the following result.

Theorem 4 *If F_1, \dots, F_k , $k \geq 2$, are connected factors of K_n , then $\gamma_t(F_1, \dots, F_k) \leq n$, and this bound is sharp.*

3.2 Minimum degree two

If we require the minimum degree of each factor to be at least two, then the upper bound in Theorem 4 can be improved. The following result is well-known (see, for example, [1, p.81]).

Theorem 5 *For any graph $G = (V, E)$, $\beta(G) \geq \sum_{v \in V} 1/(\deg(v) + 1)$.*

Corollary 6 *For any graph G of order n and size m , $\beta(G) \geq n^2/(2m + n)$.*

Using Corollary 6, we can establish the following result.

Theorem 7 *If F_1, \dots, F_k are factors of K_n with $\delta(F_i) \geq 2$ for $i = 1, \dots, k$, then $\gamma_t(F_1, \dots, F_k) \leq 2kn/(2k + 1)$, and this is sharp.*

PROOF. Let H be the combined hypergraph of the ONH of each F_i . Form the graph H' from H by replacing each hyperedge of size 3 or more by a two-element subset. Then H' is a (multi-) graph with n vertices and kn edges. A vertex cover of H' is a FTDS and so $\gamma_t(F_1, \dots, F_k) \leq \alpha(H')$. By Corollary 6, $\alpha(H') \leq n - n^2/(2kn + n) = 2nk/(2k + 1)$, as required.

Sharpness occurs whenever each component of H is isomorphic to K_{2k+1} . For example, if $n = 2k + 1$, then for $1 \leq i \leq k$, let F_i be the two-factor where each vertex v_j is adjacent to $v_{j \pm i}$ (addition modulo $2k + 1$). It is easy to see that $2k$ vertices are needed for a FTDS: any two vertices have a common neighbor in one of the factors. \square

Next we consider the case where each factor is 2-regular. We show that if at least one of the k factors contains neither a $(2k + 1)$ -cycle nor a $2(2k + 1)$ -cycle, then the upper bound in Theorem 7 can be improved.

Theorem 8 *Let F_1, \dots, F_k , $k \geq 2$, be 2-regular factors of K_n . If at least one factor contains neither a $(2k + 1)$ - nor a $2(2k + 1)$ -cycle, then $\gamma_t(F_1, \dots, F_k) \leq (2k - 1)n/(2k)$.*

PROOF. Let H be the combined hypergraph of the ONH of each F_i . Then H is a $(2k)$ -regular (multi-)graph. A vertex cover of H is a FTDS and so $\gamma_t(F_1, \dots, F_k) = \alpha(H)$. If H contains a component K_{2k+1} , then the component is formed from k $(2k + 1)$ -cycles, one from each factor. A sufficient condition for a factor not to have a $(2k + 1)$ -cycle in its ONH is that the factor contains neither a $(2k + 1)$ - nor a $2(2k + 1)$ -cycle. By assumption, this holds for at least one factor. We deduce therefore that H contains no component K_{2k+1} . Hence,

since H is $(2k)$ -regular, it is $(2k)$ -colorable by Brooks' theorem [4]. Thus H has independence number at least $n/2k$ and therefore vertex cover number at most $n(2k - 1)/(2k)$. \square

Even if one insists that the 2-regular factors be connected, one cannot improve on the above bound.

Corollary 9 *For $n \geq 3$, if F_1 and F_2 are two n -cycles with the same vertex set, then $\gamma_t(F_1, F_2) \leq 3n/4$, except if $n = 5$ or $n = 10$ (when $\max \gamma_t(F_1, F_2) = 4n/5$), and this is sharp for infinitely many n .*

PROOF. The upper bound is from Theorem 8. It remains for us to show sharpness. For $n = 5$, let F_2 be the complement of F_1 (and so the two cycles do not share an edge). Then, $\gamma_t(F_1, F_2) = 4 = 4n/5$. For $n = 10$, let $V = \{v_0, v_1, \dots, v_9\}$, and let F_1 be the 10-cycle with edges v_i, v_{i+1} and F_2 the 10-cycle with edges v_i, v_{i+3} (arithmetic modulo 10). Then, $H = 2K_5$ and $\gamma_t(F_1, F_2) = \alpha(H) = 8$.

In general, define $H = I(F)$ to be the *inflation* of a 4-regular graph F : the line graph of the subdivision graph of F . It follows that the vertex set of $I(F)$ can be partitioned into subsets of size 4 such that each subset is a clique. Now it is easy to see (and known) that an (edge) decomposition of F into two cycles extends to such a decomposition of $I(F)$. In particular, if the order m of F is even, and it can be decomposed into two subgraphs each of which is the union of two $m/2$ cycles, then $I(F)$ can be decomposed into two subgraphs each of which is the union of two $n/2$ cycles. Furthermore, any pair of $(n/2)$ -cycles is realizable as the ONH of some n -cycle. Thus $I(F)$ is the combined ONH of two n -cycles. Clearly $I(F)$ has independence number $n/4$ and hence vertex cover number $3n/4$. This provides examples of equality for n a multiple of 8.

In particular here is an explicit example where the factors are edge-disjoint. For $x = 0, \dots, n/8 - 1$, let P_x be the path $8x, 8x+7, 8x+4, 8x+1, 8x+10, 8x+5, 8x+14, 8x+11$ where addition is taken modulo n . Let $F_1: 0, 1, \dots, n-1, 0$ be the cycle with vertices numbered consecutively and let F_2 be the cycle $P_0, \dots, P_{n/8-1}, 0$. Then in H , the set $\{x, x+2, x+4, x+6\}$ is a clique for $x \equiv 0, 1 \pmod{8}$, and so $\gamma_t(F_1, F_2) = \alpha(H) = 3n/4$. \square

3.3 Higher minimum degree

If we require the minimum degree of each factor to be at least three, then the upper bound in Theorem 7 can be improved. Surprisingly perhas, the best

bounds for minimum degree 3 and minimum degree 4 come from two different sources.

There are several upper bounds for the transversal number of a hypergraph in terms of the number of vertices n and number of hyperedges m : see for example, [5,13]. In general they are not optimal for $m \gg n$, which is the case we need. Instead, the best general bound we know uses the obvious extension to the standard probabilistic-method upper bound for domination number (see [1]):

Theorem 10 *For an r -uniform hypergraph H with n vertices and m hyperedges, $\tau(H) \leq n(\ln(rm/n) + 1)/r$.*

PROOF. Construct a transversal T as follows. Take each vertex independently with probability p . Then for each missed hyperedge, take one vertex in that hyperedge.

The expected number of hyperedges missed is $m(1-p)^r$. By linearity of expectation, it follows that $|E(T)| \leq np + m(1-p)^r$. Thus $|E(T)| \leq np + me^{-pr}$. Set $p = \ln(rm/n)/r$. Then $E(T) \leq n(\ln(rm/n) + 1)/r$, as required. \square

It follows:

Theorem 11 *Let F_1, F_2, \dots, F_k be factors of K_n . Let $\delta = \min\{\delta(F_i) \mid i = 1, \dots, k\}$. Then $\gamma_t(F_1, F_2, \dots, F_k) \leq (\ln \delta + \ln k + 1)n/\delta$.*

PROOF. The combined ONH has n vertices and (at most) kn hyperedges. If a hyperedge is bigger than δ , simply shrink it; so we may assume it is δ -uniform. The result follows since the bound of Theorem 10 is increasing in m . \square

For small examples, one can do the exact optimization of the upper bound $np + m(1-p)^r$ in the proof of Theorem 10. For example, if $m = 2n$ and $r = 3$, then $p^* = 1 - 1/\sqrt{6}$ and $|E(T)| \leq n(1 - \sqrt{2/27})$. Thus:

Corollary 12 *If F_1 and F_2 are factors of the complete graph K_n with $\delta(F_i) \geq 3$ for $i = 1, 2$, then $\gamma_t(F_1, F_2) \leq n(1 - \sqrt{2/27}) \approx 0.728n$.*

If we require the minimum degree of each factor to be at least four, then the best upper bound we know is from a different hypergraph bound:

Theorem 13 (Thomasse, Yeo [13]) *If H is a 4-uniform hypergraph with n*

vertices and m hyperedges, then $\tau(H) \leq (5n + 4m)/21$.

This yields the following result.

Corollary 14 *If F_1 and F_2 are factors of the complete graph K_n with $\delta(F_i) \geq 4$ for $i = 1, 2$, then $\gamma_t(F_1, F_2) \leq 13n/21$.*

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