Double Vertex Graphs and Complete Double Vertex Graphs

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Abstract

Let $G = (V, E)$ be a graph of order $n \geq 2$. The double vertex graph, $U_2(G)$, is the graph whose vertex set consists of all $\binom{n}{2}$ unordered pairs from V such that two vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \bigcap \{u, v\}| = 1$ and if $x = u$, then y and v are adjacent in G. A generalization is called a complete double vertex graph, denoted by $CU_2(G)$, and is similar to $U_2(G)$, except that the vertex set is all $\binom{n+1}{2}$ unordered 2-multisets of elements of V. We look at some properties of these two graph products and investigate the problem of reconstructing a graph from its (complete) double vertex graph.

1 Introduction

There are many graph functions with which one can construct a new graph from a given graph, such as the Cartesian product and the line graph. One such graph function is called the *double vertex graph*. This was introduced by Alavi et al. in [2], and studied in [1, 3] inter alia. For a survey, see [6].

Let $G = (V, E)$ be a graph with order $n \geq 2$. The *double vertex graph*, denoted by $U_2(G)$, is the graph whose vertex set consists of all $\binom{n}{2}$ unordered pairs of V such that two vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \bigcap \{u, v\}| = 1$ and if $x = u$, then y and v are adjacent in G. An example of a double vertex graph is given in Figure 1.

We introduce a natural generalization of this concept called the *complete double vertex graph*. This product was implicitly introduced by Chartrand et al. in [7], and used in [8]. The *complete double vertex graph* of G, denoted by $CU_2(G)$, is the graph whose vertex set consists of all $\binom{n+1}{2}$ unordered pairs of elements of V (duplicates allowed). That is, it contains all the vertices of $U_2(G)$ and all 2-element multisets of the form $\{a, a\}$. Again two vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \bigcap \{u, v\}| = 1$

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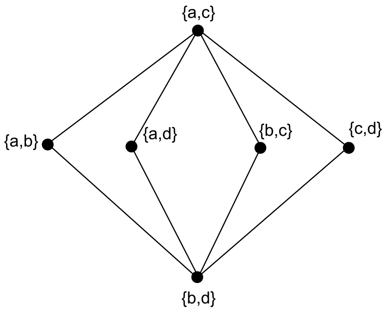

Figure 1: The double vertex graph of a 4-cycle, *abcda*

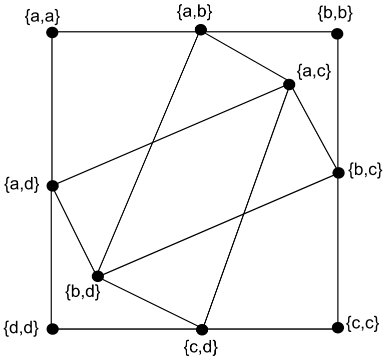

Figure 2: The complete double vertex graph of a 4-cycle, *abcda*

and if $x = u$, then y and v are adjacent in G. Figure 2 gives an example of a complete double vertex graph.

In this paper we look at some properties of double vertex graph and complete double vertex graph. We also investigate the problem of reproducing the original graph from these two graph products.

2 Basics of Double Vertex Graphs

We review the basic properties of double vertex graphs which will help us to study the problem of reconstructing G from $U_2(G)$.

Observation 2.1. [2] *If G has n vertices and m edges, then the double vertex graph of G has $n(n-1)/2$ vertices and $m(n-2)$ edges.*

Indeed for each edge of G there are $n-2$ edges of $U_2(G)$.

Observation 2.2. [2] *The degree of the vertex $\{x, y\}$ in $U_2(G)$ is:*

(i) $\deg_G(x) + \deg_G(y)$, *if $xy \notin E(G)$,*

(ii) $\deg_G(x) + \deg_G(y) - 2$, *otherwise.*

Corollary 2.3. [2] *If G is a connected graph, then $U_2(G)$ is regular if and only if G is either a complete graph or $K(1,3)$.*

Theorem 2.4. [2, 6] *a) $U_2(G)$ is a tree if and only if $G = K_2$ or $G = P_3$. b) $U_2(G)$ is a cycle if and only if $G = K_3$ or $G = K(1,3)$.*

c) If G is connected, $U_2(G)$ is Eulerian if and only if the degree of all vertices in G have the same parity.
d) $U_2(K_n)$ is the line graph of K_n.

Theorem 2.5. [2] *G is connected if and only if $U_2(G)$ is connected. Indeed, if G has k components each of order at least two, then $U_2(G)$ has $k(k+1)/2$ components.*

Theorem 2.6. [6, 10] *If G is a k-connected graph with $k \geq 3$, then $U_2(G)$ is $(2k-2)$-connected.*

Since $U_2(G)$ is a subgraph of the Cartesian product $G \square G$, it follows that:

Observation 2.7. *If G is k-colorable, then $U_2(G)$ is k-colorable.*

Theorem 2.8. *If G contains k triangles, then $U_2(G)$ contains $k(n-2)$ triangles.*

Proof. For any triangle abc in G, the vertices $\{a, b\}$, $\{b, c\}$ and $\{a, c\}$ form a triangle in $U_2(G)$. Also for any $d \neq a, b, c$, the vertices $\{d, a\}$, $\{d, b\}$ and $\{d, c\}$ form a triangle in $U_2(G)$. Thus there are at least $k(n-2)$ triangles in $U_2(G)$.

On the other hand, consider any triangle T' in $U_2(G)$. Say two of its vertices are $\{a, b\}$ and $\{b, c\}$. Then the third vertex is either $\{a, c\}$ or $\{b, e\}$ for some e. It follows that T' has one of the above forms. $\square$

In particular, $U_2(G)$ has a triangle if and only if G has one.

For more results on double vertex graphs, see [1, 2, 3, 6].

3 Properties of $CU_2(G)$

The complete double vertex graph of G, $CU_2(G)$, is similar to the double vertex graph of G. We explore its properties next.

Observation 3.1. *If G has n vertices and m edges, then $CU_2(G)$ has $n(n+1)/2$ vertices and nm edges.*

Again, if G is empty then so is $CU_2(G)$.

Observation 3.2. *Let x and y be two vertices of a graph G. Then the degree of the vertex $\{x, y\}$ of the complete double vertex graph is*

(i) $\deg_G(x)$, *if $x = y$, and*

(ii) $\deg_G(x) + \deg_G(y)$, *otherwise.*

Proof. The pair $\{x, x\}$ in $CU_2(G)$ is adjacent to only the pairs $\{x, a\}$ where a is adjacent to x in G. The pair $\{x, y\}$ with $x \neq y$ in $CU_2(G)$ is adjacent to all the $\{x, a\}$ where a is adjacent to y in G and to all the $\{y, b\}$ where b is adjacent to x in G. $\square$

Corollary 3.3. *If the graph $CU_2(G)$ is regular, then it is empty.*

Proof. Assume $CU_2(G)$ is regular. Let x and y be distinct vertices in G. Then the pairs $\{x, x\}$, $\{y, y\}$ and $\{x, y\}$ all have the same degree in $CU_2(G)$. This can occur only if $\deg x = \deg y = 0$. $\square$

For example, $CU_2(G)$ is not a cycle.

Theorem 3.4. *The graph $U_2(G)$ is an induced subgraph of $CU_2(G)$, and $CU_2(G)$ is an induced subgraph of the Cartesian product $G \square G$. The graph G is an induced subgraph of $CU_2(G)$. Indeed, the edges of $CU_2(G)$ can be partitioned into n sets such that each set induces a copy of G.*

For example, if G contains a cycle of length r then $CU_2(G)$ contains a cycle of length r.

As a consequence we get:

Theorem 3.5. *The chromatic number of $CU_2(G)$ is the same as the chromatic number of G.*

Proof. $CU_2(G)$ contains a copy of G but is a subgraph of $G \square G$. $\square$

For example, $CU_2(G)$ is bipartite if and only if G is bipartite.

It is easy to see:

Theorem 3.6. *G is connected if and only if the complete double vertex graph of G is connected. Indeed, if G has k components, then $CU_2(G)$ has $k(k+1)/2$ components.*

Corollary 3.7. *Let G be a connected graph. The graph $CU_2(G)$ is Eulerian if and only if $\deg_G(v)$ is even $\forall v \in V(G)$.*

Theorem 3.8. [8] *If G is k-connected, then so is $CU_2(G)$.*

Theorem 3.9. *The complete double vertex graph of G is a tree if and only if $G = K_1$ or K_2.*

Proof. The graph $CU_2(P_3)$ contains a cycle. Hence, if $CU_2(G)$ is a tree then G is K_1 or K_2. These have complete double vertex graphs K_1 and P_3, respectively. $\square$

4 Planarity

Alavi et al. determined when the double vertex graph of a connected graph is planar.

Theorem 4.1. [1] *Let G be a connected graph. The graph $U_2(G)$ is planar if and only if G is either a path or a connected subgraph of any of the six graphs shown in Figure 3.*

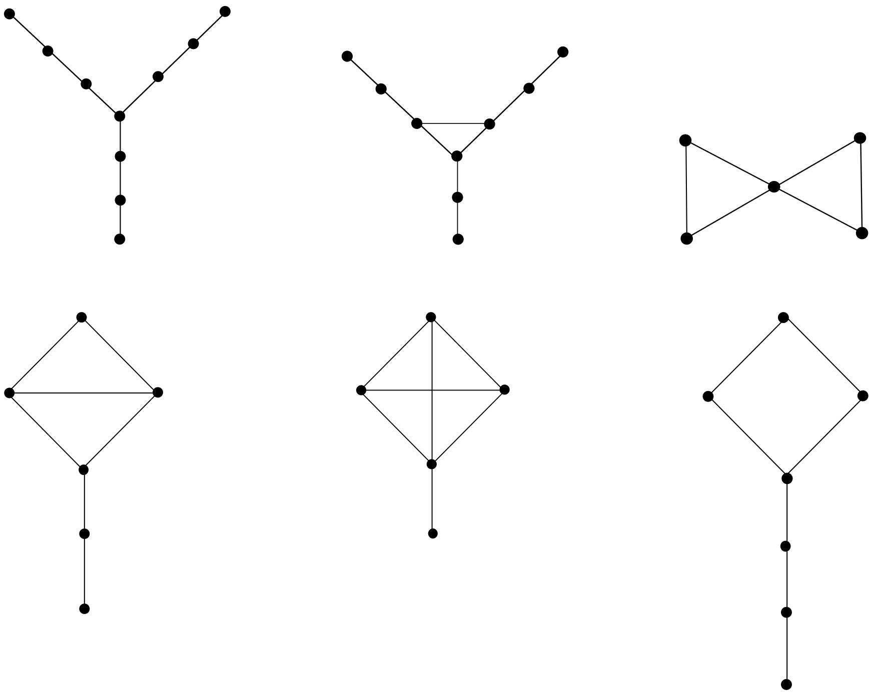

Figure 3: Graphs whose double vertex graphs are planar

A similar result holds for complete double vertex graphs:

Theorem 4.2. *Let G be a connected graph. The graph $CU_2(G)$ is planar if and only if either G is a path or a connected subgraph of any of the five graphs shown in Figure 4.*

A sketch for the proof is given below.

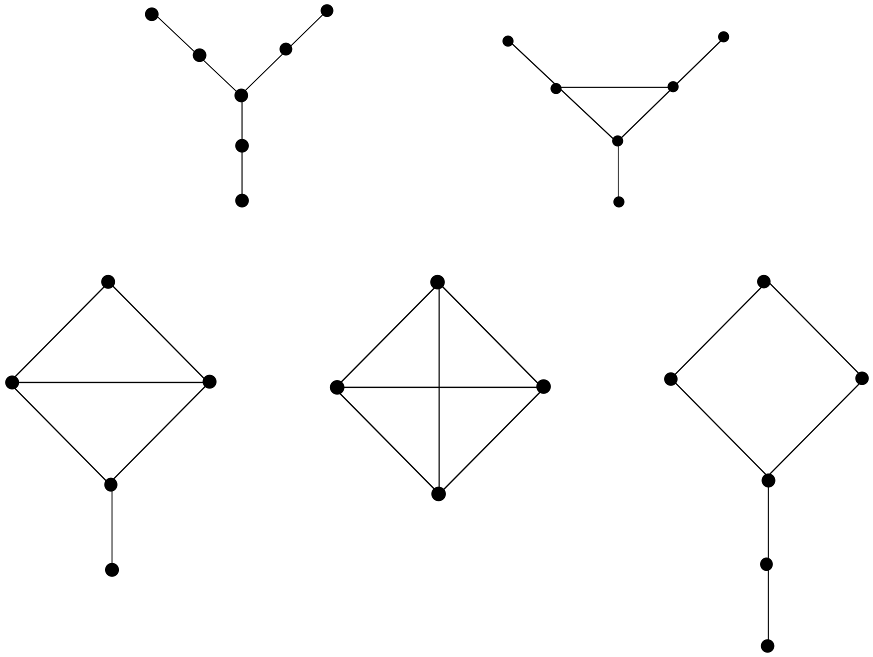

Figure 4: Graphs whose complete double vertex graphs are planar

(⇒) If G is one of the five graphs in Figure 4 or a path then by construction, $CU_2(G)$ is planar.

(⇐) Now assume that $CU_2(G)$ is planar. We know that $U_2(G)$ is an induced subgraph of $CU_2(G)$ and hence $U_2(G)$ is planar. Thus by Theorem 4.1, G is a path or a subgraph of one of the six graphs shown in Figure 3. If G is a path, then we are done. If G is not a path, then one can easily check that the maximal subgraphs of the graphs in Figure 3 whose complete double vertex graphs are planar are listed in Figure 4.

5 Hamiltonian Properties

The following results about Hamiltonicity have been obtained.

Theorem 5.1. [4] *For $n = 4$ or $n \geq 6$, $U_2(C_n)$ is not Hamiltonian.*

A cycle with an odd chord is a graph obtained by adding the edge $1k$ to C_n, where k is odd.

Theorem 5.2. [5, 6] *Let G be a Hamiltonian graph of order $n \geq 4$. Then $U_2(G)$ is Hamiltonian if and only if a Hamiltonian cycle of G has an odd chord or if $n = 5$.*

We believe similar results hold for the complete double vertex graph. We provide a result for a specific chord in Theorem 5.4.

Theorem 5.3. *For $n \geq 4$, $CU_2(C_n)$ is not Hamiltonian. In fact it is not 1-tough.*

Proof. Let the vertices of C_n be labeled $1, 2, \ldots, n$ and let $S = \{\{1, 2\}, \{2, 3\}, \ldots, \{n, 1\}\}$. Then the graph $CU_2(C_n) - S$ has at least n isolated vertices, since the neighbors of the vertices $\{i, i\}$, $1 \leq i \leq n$, in $CU_2(C_n)$ are contained in S. Thus $CU_2(C_n)$ has at least $n+1$ components, if $n \geq 4$. Thus, $CU_2(C_n)$ is not Hamiltonian for $n \geq 4$. $\square$

Theorem 5.4. *Let G be a cycle on n vertices. Let G' be obtained from G by adding a chord between two vertices of G having distance two between them. Then $CU_2(G')$ is Hamiltonian.*

Proof. **Case 1:** n is odd.

The idea is that $CU_2(C_n)$ has a spanning 2-factor when n is odd, and edges that correspond to the chord will serve as bridges between the factors.

Wlog, asssume the vertices of G are numbered 0 to $n-1$ and let the chord added to get G' be 02. Let $H = CU_2(C_n)$. For any vertex $\{i, j\}$ in $CU_2(G)$, the distance $d(i, j)$ between i and j in G is in the range $0 \leq d \leq (n-1)/2$. Construct a spanning 2-factor for H as follows:

For $0 \leq i \leq \lfloor \frac{n-3}{4} \rfloor$, let S_i be the graph induced by vertices of the form $\{u, v\}$ such that $d(u, v) \in \{2i, 2i+1\}$. Each S_i is a cycle on $2n$ vertices, and if $n \equiv 3 \mod 4$, then $S = \{S_i \mid 0 \leq i \leq \lfloor \frac{n-3}{4} \rfloor\}$ form a spanning 2-factor. If $n \neq 4r+3$ for some non-negative integer r, then let $j = \lfloor \frac{n-3}{4} \rfloor + 1$ and S_j be the cycle induced by the n vertices of the form $\{u, v\}$ such that $d(u, v) \in \{2j, 2j+1\}$. Thus, when $n \neq 4r+3$ for some non-negative integer r, $S \cup S_j$ forms a spanning 2-factor of H.

Now, let $H' = CU_2(G')$. Adding the chord to G adds n edges to H; call each a *bridge-edge*. In each cycle S_i except for $i \geq (n-3)/4$ there are two consecutive vertices $\{0, -2i\}$ and $\{0, -2i-1\}$ (arithmetic modulo n), and two bridge-edges joining these to $\{2, -2i\}$ and $\{2, -2i-1\}$, which are consecutive in cycle S_{i+1}. We use these bridge-edges to go between cycles to form a Hamiltonian cycle of H'.

A Hamiltonian cycle obtained using this idea for C_{11} with the chord 02 is given in Figure 5.

Case 2: n is even.

Wlog, let G be a cycle of the form $\{0, n-1, 1, 2, \ldots, n-2\}$ and G' be obtained by adding the chord 01. Consider $C = G' - \{n-1\}$. Now C is a cycle on $n-1$ vertices where n is even and hence as in Case 1 we can find a spanning 2-factor S for $CU_2(C)$. Also the subgraph of $CU_2(G')$ induced by $S' = \{\{0, n-1\}, \{1, n-1\}, \ldots \{n-1, n-1\}\}$ is a cycle on n vertices and $S \cup S'$ is a spanning 2-factor for $H' = CU_2(G')$.

For each $S_i \in S$, the consecutive vertices $\{0, 2i\}$ and $\{0, 2i+1\}$ are adjacent to the consecutive vertices $\{2i, n-1\}$ and $\{2i+1, n-1\}$ respectively in S'. Hence we can form a Hamiltonian cycle in $H' = CU_2(G')$. $\square$

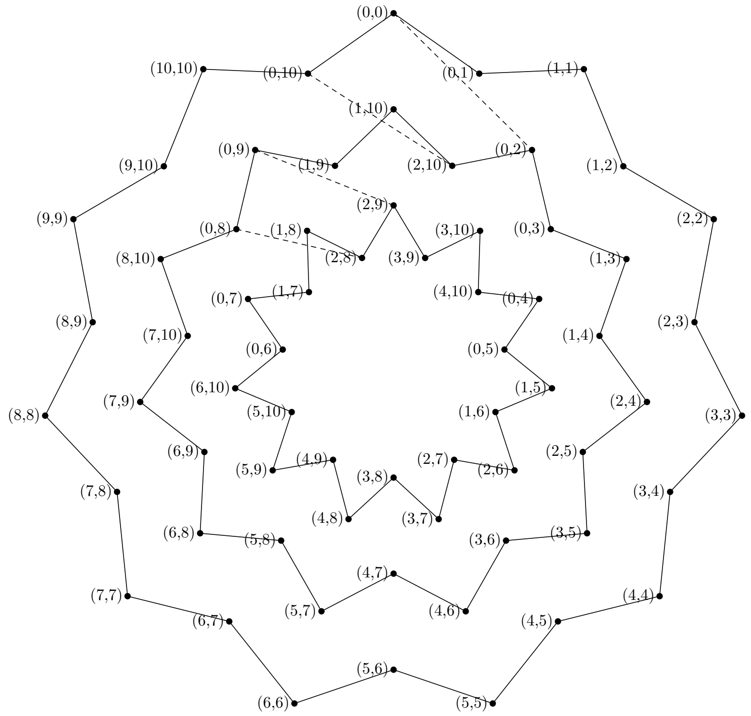

Figure 5: A Hamiltonian cycle in $CU_2(G)$ where $G = C_{11}$ with the chord 02.

6 Reconstruction of G from $CU_2(G)$ and $U_2(G)$

One of the major challenges in the study of graph products is to reproduce the original graph from the graph product. In this section we examine some more properties of these graph products which help one to reconstruct some classes of graphs from $CU_2(G)$ and $U_2(G)$.

Note that reconstructing the graph G from the Cartesian product $G \Box G$ is solved. But the techniques do not seem to be applicable here. For more details on reconstructing a graph from its Cartesian product see [9].

6.1 Reconstructing G from $CU_2(G)$

We start with the complete double vertex graph case. We call the vertices of the form $\{x, x\}$ *twin-pairs.*

Theorem 6.1. *Let G be a graph. Then $xy \in E(G)$ if and only if the twin-pairs $\{x, x\}$ and $\{y, y\}$ have a common neighbor in $CU_2(G)$.*

Proof. The only possible vertex that could be a common neighbor of the pairs $\{x, x\}$ and $\{y, y\}$ in $CU_2(G)$ is the pair $\{x, y\}$. This is a common neighbor if and only if $xy \in E(G)$. $\square$

Corollary 6.2. *If one can identify the twin-pairs of $CU_2(G)$, then one can construct the line-graph of G. Hence one reconstruct the graph G.*

Corollary 6.3. *If G is either regular or the degree of every vertex in G is odd, then one can reconstruct G from $CU_2(G)$.*

Proof. If all vertices of G are of degree r then the degree of twin-pairs are r and that of the non-twin-pairs are $2r$. So one can identify the twin-pairs and reconstruct G.

If the vertices of G are of odd degree, then the twin-pairs of $CU_2(G)$ have odd degree, while any other vertex of $CU_2(G)$ has even degree. So one can identify the twin-pairs and hence reconstruct G. $\square$

6.2 Reconstructing G from $U_2(G)$

We next consider the double vertex graph case. We call $\{a, b\} \in V(U_2(G))$ a *line-pair* if and only if $ab \in E(G)$. Hence each vertex of a double vertex graph is either a line-pair or a non-line-pair.

Theorem 6.4. *Two line-pairs in $U_2(G)$ are adjacent if and only if the corresponding edges lie in a triangle.*

Proof. Assume line-pairs $\{a, b\}$ and $\{a, c\}$ are adjacent. By the definition of double vertex graph, $bc \in E(G)$, and hence ab, ac, bc form a K_3.

If ab, ac are edges in a K_3 in G, then by the definition of $U_2(G)$, the pairs $\{a, b\}$ and $\{a, c\}$ will be adjacent. ☐

Theorem 6.5. *Two line-pairs in* $U_2(G)$ *have a common neighbor in* $U_2(G)$ *if and only if the corresponding edges are either adjacent in* G *or lie in a* 4*-cycle of* G*.*

Proof. ($\Leftarrow$) If edges ab and bc are adjacent, then $\{a, c\}$ is a common neighbor to $\{a, b\}$ and $\{b, c\}$ in $U_2(G)$. If edges ab and cd lie in a 4-cycle of G, say $abcda$, then $\{a, b\}$ and $\{c, d\}$ have common neighbors $\{a, c\}$ and $\{b, d\}$.

($\Rightarrow$) Suppose two line-pairs $\{a, b\}$ and $\{x, y\}$ have a common neighbor in $U_2(G)$. If the two line-pairs overlap, say $a = x$, then clearly the corresponding edges are adjacent. If the line-pairs don't overlap, then the common neighbor has one element from $\{a, b\}$ and one element from $\{x, y\}$. Say the common neighbor is $\{a, x\}$. Then, $abyxa$ forms a 4-cycle in G. ☐

Corollary 6.6. *Suppose* G *has no* 4*-cycle. If one can identify the line-pairs in* $U_2(G)$*, then one can construct the line graph of* G *and hence one can reconstruct* G*.*

Note that the line graphs of K_3 and $K_{1,3}$ are isomorphic. However, one can distinguish $U_2(K_3)$ from $U_2(K_{1,3})$ by the number of vertices.

Corollary 6.7. *If* G *is regular and has no* 4*-cycle, then one can reconstruct* G *from* $U_2(G)$*.*

Proof. The line-pairs of $U_2(G)$ have degree 2 less than the non-line-pairs, by Observation 2.2. So one can recognize them, and hence by Corollary 6.6 one can reconstruct G. ☐

6.3 Reconstructing cubic graphs

The presence of 4-cycles in G seems to make the reconstruction a little harder. To overcome this, we restrict our attention to 3-regular graphs.

Theorem 6.8. *Let* G *be a cubic graph. The corresponding edges of two line-pairs lie in an induced* 4*-cycle in* G *if and only if the line-pairs lie in an induced* $K_{2,4}$ *in* $U_2(G)$ *with the* 4 *line-pairs as one partite set and the* 2 *non-line-pairs as the other partite set.*

Proof. ($\Rightarrow$) By the definition of $U_2(C_4)$.

($\Leftarrow$) Consider an induced $H = K_{2,4}$ as in the hypothesis. Let a be any vertex in a line-pair of H. Then a cannot occur in all 4 line-pairs, since G is cubic.

Suppose a occurs in 3 line-pairs; say $\{a, b\}, \{a, c\}, \{a, d\}$. Then a must occur in both the non-line-pairs of H; say $\{a, x\}$ and $\{a, y\}$. Then the fourth line-pair is $\{x, y\}$. But then x is adjacent to all of b, c, d and y in G, which is a contradiction. It follows that a occurs in at most 2 line-pairs. Let $\{a, b\}$ be such a line-pair.

Consider a non-line-pair of H; say the pair $\{a, x\}$. Then x is in at most two line-pairs, by the previous paragraph. It follows that vertices a and x lie in exactly two line-pairs of H each, because all four line-pairs of H are adjacent to $\{a, x\}$. Also, if the other non-line-pair contains a or x, then it contains the other one too, a contradiction. So the two non-line-pairs do not overlap. Wlog, say the other non-line pair is $\{b, y\}$.

Then the line-pairs are $\{a, b\}$, $\{a, y\}$, $\{b, x\}$ and $\{x, y\}$. These four line-pairs induce a 4-cycle in G. Hence the proof. $\square$

So, in a cubic graph one can identify the line-pairs, and hence one can identify the induced 4-cycles as well as the non-induced 4-cycles. The idea is to construct the line graph, except that at this point in some cases one can only identify the 4 vertices which form a cycle, without knowing the order of the vertices .

Theorem 6.9. *Let G be a cubic graph. Suppose two line-pairs lie in $K_{2,4}$ representing an induced C_4 in G, but not in a second $K_{2,4}$. Then the two line-pairs are adjacent if and only if they have a common line-pair at a distance 2 which does not lie in the same $K_{2,4}$.*

Proof. ($\Rightarrow$) Suppose ab and ac lie in an induced 4-cycle in G. Since G is a cubic graph, there exists $ad \in E(G)$. In $U_2(G)$, the line-pairs $\{a, b\}$ and $\{a, c\}$ have $\{a, d\}$ as a common neighbor at a distance 2.

($\Leftarrow$) Suppose that two distinct line-pairs $\{a, b\}$ and $\{x, y\}$ have a line-pair as a common neighbor at distance 2. If the elements of the common neighbor are distinct from the line-pairs, say $\{c, d\}$, then there is a 4-cycle in G containing either ab or xy, and cd. In any case, we get a contradiction, as we assumed that the line-pairs $\{a, b\}$ and $\{x, y\}$ do not lie in a second $K_{2,4}$. Also, the common distance-2 neighbor cannot overlap with both line-pairs $\{a, b\}$ and $\{x, y\}$, because if it does then it will be in the same $K_{2,4}$ as $\{a, b\}$ and $\{x, y\}$. If the common distance-2 neighbor overlaps with one line-pair, say $\{a, b\}$, then $\{a, b\}$ will be part of a second $K_{2,4}$.

Hence if we assume that $\{a, b\}$ and $\{x, y\}$ do not overlap, then we get a contradiction and hence the proof. $\square$

One can therefore determine the order of the edges in the case of 4-cycles. If two 4 cycles overlap then one can identify the overlapping $K_{2,4}$ and hence determine the overlapping edges.

Corollary 6.10. *Given $U_2(G)$, one can reconstruct G if any of the following is true.*

(i) G is a cubic graph

(ii) G is regular and contains no 4-cycles

(iii) G contains no 4-cycles and one can identify the line-pairs in $U_2(G)$

6.4 Reconstruction of trees

In [2], the authors made the following observation and stated Theorem 6.12.

Observation 6.11. [2] *If a connected graph H whose order is at least three is the double vertex graph of some graph G of order n, then G has a vertex of degree one if and only if H contains $n-2$ independent edges whose removal from $H = U_2(G)$ results in exactly two components, one of which is $G - v$ and the other, $U_2(G - v)$.*

Theorem 6.12. [2] *Let T and T' be trees. Then $U_2(T) \cong U_2(T')$ if and only if $T \cong T'$.*

However, [2] does not provide full proofs for these results. As regards to complete double vertex graphs, we are able to prove one direction of the equivalent result.

Theorem 6.13. *Let $H = CU_2(G)$ be connected. Then there exist $(n-1)\delta(G)$ edges whose deletion will result in a graph with more than one component, one of which is isomorphic to G.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Note that for any vertex v_i of G, the subgraph of H induced by $S = \{\{v_i, v_1\}, \{v_i, v_2\}, \ldots \{v_i, v_n\}\}$, denoted by $H[S]$, is isomorphic to G and $H - S$ is isomorphic to $CU_2(G - v)$.

Wlog, assume $v = v_1$ and let $\deg_G(v) = \delta = \delta(G)$. Let $S = \{\{v, v_i\} \mid 1 \leq i \leq n\}$. The vertex $\{v, v_i\}$, $2 \leq i \leq n$, is adjacent to exactly δ vertices not in S. Let M be the set of edges joining the vertices in S to the vertices not in S. Clearly, $|M| = (n-1)\delta$ and $H - M$ has more than one component, one of which is the graph induced by S which is isomorphic to G. $\square$

We are unable to show that one can reconstruct trees from their complete double vertex graphs. However, we have verified by computer up to order 10 that different trees give different complete double vertex graphs.

7 Open Questions

(i) Is it possible to develop an algorithm to reproduce G from $U_2(G)$ and $CU_2(G)$ for all classes of graphs? Is it in fact true that if G and H are different graphs then $U_2(G)$ and $U_2(H)$ are non-isomorphic? How about $CU_2(G)$ and $CU_2(H)$?

(ii) What can one say about the domination properties of $U_2(G)$ and $CU_2(G)$?

(iii) One can naturally extend these concepts to the directed graph version. What can one say about the properties of the directed version?

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