On Distances between Isomorphism Classes of Graphs

Gerhard Benadé, Johannesburg
Wayne Goddard, Cambridge, USA
Terry A. McKee, Dayton
Paul A. Winter, Durban

Summary

In 1986, Chartrand, Saba and Zou [3] defined a measure of the distance between (the isomorphism classes of) two graphs based on 'edge rotations'. Here, that measure and two related measures are explored. Various bounds, exact values for classes of graphs and relationships are proved, and the three measures are shown to be intimately linked to 'slowly-changing' parameters.

AMS Classification: 05C99

Key Words: graphs, distance, deformations

1 Introduction

Various measures of distance between (the isomorphism classes of) two graphs have been proposed. These include measures proposed by Zelinka [6],[7], Baláž, Koča, Kvasníčka & Sekanina [1] and Johnson [4]. Further, there is the edge rotation distance of Chartrand, Saba & Zou [3] and the edge shift distance of Johnson [5]. It is on the latter two that we build. For terminology and notation not explained here, see [2].

We specifically consider measures based on the idea of *deformations* which translate a graph G to a graph G'. Examples of deformations that might be considered are the removal or addition of an edge or vertex, or a combination thereof. We say that a graph G can be *transformed* into a graph H if there exists a finite sequence of deformations which cumulatively translate G into H. If G can be transformed into H, then the minimum number of deformations needed to transform G into H is the *distance* from G to H.

Here we are concerned with deformations of the type $G' = G - e_1 + e_2$ where $e_1 \in E(G)$ and $e_2 \in E(\bar{G})$. ($E(G)$ denotes the set of edges of G, and $\bar{G}$ the complement of G.) Note that this deformation is reversible, i.e. G' may be deformed into G by a similar deformation. The two restricted classes we shall deal with also are reversible. It is evident that the relation 'can be transformed into' is thus an equivalence relation on the set of all graphs, and that the associated distance function will yield a metric space on each of the equivalence classes.

When there is no (further) restriction on e_1 and e_2 we obtain the *edge move* distance which we shall denote by $\delta_M(G, H)$. One may think of 'moving' e_1 to e_2, hence the name. If one prescribes that e_1 and e_2 have a vertex in common, then one obtains the *edge rotation* distance of Chartrand, Saba and Zou [3] which we may denote by $\delta_R(G, H)$.

If one further prescribes that, not only do $e_1 = uv$ and $e_2 = xy$ have an end-vertex in common (say $x = u$), but also that their other end-vertices are adjacent (i.e. $vy \in E(G)$), then one obtains what we shall call the *edge slide* distance. This was introduced by Johnson [5] as the edge shift distance. We shall denote this by $\delta_S(G, H)$. Our main focus will be on edge slide distance, though we also look at the other two measures.

2 Some General Ideas

An immediate relationship from the above definitions is that if $\delta_S(G, H)$ is defined for graphs G and H, then

$$\delta_M(G, H) \leq \delta_R(G, H) \leq \delta_S(G, H).$$

We need to consider for what graphs G and H these measures are defined. It is clear that if G and H are graphs such that G may be transformed into H by edge moves, then they have the same order (number of vertices) and size (number of edges). Further, it is immediately evident that the converse implication holds for general edge moves; also, in [3] it is shown that the same holds true for edge rotations. The following lemma is relevant.

Lemma 1

1) An arbitrary edge move may be achieved by (at most) two edge rotations.
2) An edge rotation $e = uv$ to $e' = uw$ may be achieved by s edge slides where s is the distance between v and w measured in $G - e$ (or equivalently $G' - e'$).

Proof

1) If this edge move is already an edge rotation then we are done. Thus we may assume that $e = uv$ is moved to $e' = xy$, all four end-vertices being distinct. If $ux \notin E(G)$ then we may rotate e to ux and thence to xy; by a similar procedure, if any of uy, vy or vx are not edges in G then we are done. Therefore it remains only to consider the case where the graph induced by u, v, x, y is complete except for xy. But in that case, one may rotate ux to xy and then uv to ux achieving the required effect.
2) Let $P : x_0 = v, x_1, x_2, \ldots, x_{s-1}, w = x_s$ be a shortest (v, w)-path in $G - e$. There are two cases to consider.

The first case is that P does not take in u. Then let i_1 be the largest subscript such that x_{i_1} is a neighbour (in G) of u. Then slide the edge $e = ux_i$ repeatedly along P until it lies in the position uw. If $i_1 = 0$ then we are done. Otherwise, let i_2 be the second largest subscript such that x_{i_2} is a neighbour (in G) of u. Then slide the edge $e = ux_{i_2}$ along P until it lies joining u to x_{i_1}. Repeating this procedure sufficiently many times (at most thrice as by the choice of P it holds that $|N(u) \cap V(P)| \leq 3$), achieves the desired edge rotation. In any implementation s slides have been used.

The second case is that P does take in u. Then by the choice of P, $u = x_i$ is adjacent (in G) only to v, x_{i-1} and x_{i+1} and to no other vertex of P; further, $x_{i-1}x_{i+1} \notin E(G)$. Then the rotation may be achieved by the following sequence of slides: slide ux_{i-1} to $x_{i-1}x_{i+1}$; slide ux_{i+1} along P to uw; slide uv along P to ux_{i-1}, and finally slide $x_{i-1}x_{i+1}$ to ux_{i+1}. In all, s slides are used, and the proof of the lemma is complete. □

Corollary 1 [3] *Let G and H be graphs. Then G may be transformed into H via edge rotations (and so $\delta_R(G, H)$ is defined) if and only if G and H have the same order and size.*

The situation for edge slides is slightly more complicated. Using the above lemma, one may prove the following proposition first observed in [5].

Proposition 1 [5] *Let G and H be graphs. Then G may be transformed into H via edge slides (and so $\delta_S(G, H)$ is defined) if and only if G and H have the same orders and sizes of components; i.e. G and H have the same number (k say) of components and these may be labelled $G_1, G_2, \ldots, G_k$ and $H_1, H_2, \ldots, H_k$ respectively such that G_i and H_i have the same order and size for $i = 1, 2, \ldots, k$.*

We shall return to these concepts presently.

3 Slowly-Changing Parameters

Many parameters are relatively unaffected by edge moves, rotations or slides. In this section we study several examples of this behaviour; these examples will be utilised to prove results in later sections. We say that a parameter μ is *slowly-changing* with respect to a particular deformation iff for all graphs G and deformations G' of G it holds that $|\mu(G') - \mu(G)| \leq 1$.

For edge moves in general, some parameters are slowly-changing by virtue of the fact that such parameters can be altered by at most one by the insertion or deletion of an edge. Examples include the minimum degree δ, the maximum degree Δ and the connectivity κ. Other parameters which depend more on matters of distance are (in general) slowly-changing only for edge slides.

Lemma 2 *The following parameters are slowly-changing with respect to edge slides:*
1) The distance $d_G(u,v)$ between any two fixed vertices u and v of G,
2) the diameter $\operatorname{diam}(G)$, and
3) the girth $g(G)$.

Proof
Let the graph G' be formed from the graph G by an edge slide. By symmetry, G' may be formed from G by an edge slide. The three parts follow:
1) Let P be a shortest (u,v)-path in G. If the edge e that was moved was not on P, then P is a path in G' and thus $d_{G'}(u,v) \leq d_G(u,v)$. On the other hand, if e was on P, say $e = xy$ was moved to $e' = xm$, then $my \in E(G)$

so that by replacing e in P by the path xmy one obtains a (u, v)-walk in G. Thus, in either case, $d_{G'}(u, v) \leq d_G(u, v) + 1$ and by symmetry the full statement is proved.

2) This follows immediately from part 1.

3) Let C be a shortest cycle in G. As in the proof of part 1, if the edge e that was moved was not on C then $g(G') \leq g(G)$, while if it was on C, say $e = xy$ was moved to $e' = xm$, then by removing e from C and replacing it with the path xmy one obtains a closed trail in G'. Thus, in either case $g(G') \leq g(G) + 1$ and the statement follows. □

Distance-related measures are not slowly-changing with respect to edge rotations. For example, $C_n \cup K_1$ may be formed from P_{n+1} by a single edge rotation. The former has infinite diameter while the latter's is finite.

Another parameter that may be slowly-changing involves end-vertices. Indeed, one may define

$$\begin{aligned} \mathrm{end}(G) &:= |\{\, v \in V(G) : \deg v = 1 \,\}|, \quad \text{and} \\ \mathrm{end}'(G) &:= |\{\, v \in V(G) : \deg v \leq 1 \,\}|. \end{aligned}$$

We summarise some observations in the following lemma.

Lemma 3 *The parameter* $\mathrm{end}'(G)$ *is slowly-changing for edge rotations and slides but not for edge moves, while* $\mathrm{end}(G)$ *is slowly-changing for edge slides only.*

Proof

The proof follows mainly from noting that an edge move may affect the degrees of four vertices (two up, two down) while an edge rotation (or edge slide) only two (one up, one down). Thus the lemma follows (and indeed with any fixed integer replacing the '1' of the above definitions), except to show that end(G) is slowly-changing with respect to edge slides. But this too follows easily noting that the vertex whose degree was reduced did not have degree one, and the vertex whose degree was increased does not now have degree one. □

Another relevant parameter is as follows. For graphs G and H, a *greatest common subgraph* of G and H is any graph F of maximum size that is (isomorphic to) a subgraph of both G and H. We shall denote the size of such a greatest common subgraph by $s(G, H)$. Clearly, if graph G' is formed from G by an edge move, then for any graph H, $|s(G, H) - s(G', H)| \leq 1$. Thus, for any fixed graph H, $s(G, H)$ represents a parameter that is slowly-changing with respect to edge moves.

4 Some Distance Formulas

It is possible to determine some specific formulas for distances. The proofs of these results are based mainly on the following technique which we formalise in the form of a lemma.

Lemma 4 *Let $\mathcal{G}$ be a collection of graphs and let $F \in \mathcal{G}$ be a designated element. Further, let μ be an integer-valued graphical parameter and consider a particular deformation. Then for that deformation, with distance between graphs G and H denoted by $\delta(G, H)$, it holds that:*

$$\forall G \in \mathcal{G} \colon (\, \delta(F, G) = |\mu(G) - \mu(F)| \,),$$

if the following three properties are satisfied:

P1 *The parameter μ is slowly-changing with respect to that particular deformation;*

P2 *F is the only element of $\mathcal{G}$ with that value of μ; and*

P3 *Given any $G \in \mathcal{G}$ with $\mu(G) \neq \mu(F)$ there exists a deformation (of the required type) yielding $G' \in \mathcal{G}$ such that $|\mu(G') - \mu(F)| < |\mu(G) - \mu(F)|$.*

Proof

The property **P1** establishes that $|\mu(G) - \mu(F)|$ is a lower bound, while properties **P2** and **P3** together show that the value is an upper bound for the distance. □

In practice, F will often have the minimum or maximum value of μ. We now use Lemma 4 to establish the following theorem which gives the edge slide distance from paths, stars and cycles to any other graph (necessarily connected and having the same order and size by Proposition 1). We denote by P_n, S_n and C_n, respectively the path, star and cycle on n vertices.

Theorem 1 *For all trees T and connected unicyclic graphs U of order n:*
1) $\delta_S(T, P_n) = \mathrm{diam}(P_n) - \mathrm{diam}(T)$,
2) $\delta_S(T, S_n) = \Delta(S_n) - \Delta(T)$, and
3) $\delta_S(U, C_n) = g(C_n) - g(U)$.

Proof

Consider Lemma 4. Property **P1** has already been observed for the parameters diameter, maximum degree and girth. Taking $\mathcal{G}$ to be the set of trees of order n for the first two, the set of unicyclic graphs of order n for the latter, and $F = P_n$, S_n or C_n, property **P2** is trivially verified. What remains is to verify that property **P3** holds in each case.

Consider firstly μ to be the diameter, and assume that T does not have diameter $n-1$. Let $P : x_1x_2 \ldots x_d$ be a longest path in T. (Note that in trees diametrical and longest paths coincide.) Then there exists a vertex y

not in P but adjacent to one of the (interior) vertices of P, say x_j. Form the tree T' by sliding the edge $x_{j-1}x_j$ to the position $x_{j-1}y$. Now, P' : $x_1x_2 \ldots x_{j-1}yx_j \ldots x_d$ is a path in T' and hence $\mathrm{diam}(T') > \mathrm{diam}(T)$. This combines with $\mathrm{diam}(P_n) \geq \mathrm{diam}(T')$ to imply **P3**.

Consider secondly μ to be the maximum degree and assume that T does not have maximum degree $n-1$. Let v be a vertex of T of maximum degree, and let $x_1, x_2, \ldots, x_d$ be its neighbours. Then there exists a vertex y not in $N[v]$ but adjacent to a vertex of $N(v)$, say x_j. Then form the tree T' by replacing the edge x_jy by the edge vy. This is an edge slide and the degree of v has been increased by one, so that $\Delta(T') > \Delta(T)$ and (as above) **P3** holds.

Consider finally, μ to be the girth and assume that U does not have girth n. Let $C = x_1x_2 \ldots x_gx_1$ be the (unique) cycle in U. As before, there exists a vertex y not in C but adjacent to a vertex of C, say x_j. Form the graph U' by sliding the edge $x_{j-1}x_j$ to the position $x_{j-1}y$. Now U' is unicyclic, so $g(U') > g(U)$ and (as above) **P3** holds.

Thus we have verified the properties **P1**, **P2** and **P3** in all three cases and the results follow from Lemma 4. □

We observed earlier that Δ is slowly-changing even for edge moves so that one has the following corollary (the result for edge rotation having also been proven in [8]).

Corollary 2 *For all trees T of order n:*

$$\delta_M(S_n, T) = \delta_R(S_n, T) = \delta_S(S_n, T) = n - 1 - \Delta(T).$$

The analogous result, however, does not hold for $\delta_R(P_n, T)$. The value $n - 1 - \mathrm{diam}(T)$ is only an upper bound for the distance. Consider, for example, the tree T shown in Figure 1. Clearly, $\mathrm{diam}(T) = 5$ and $n = 10$,

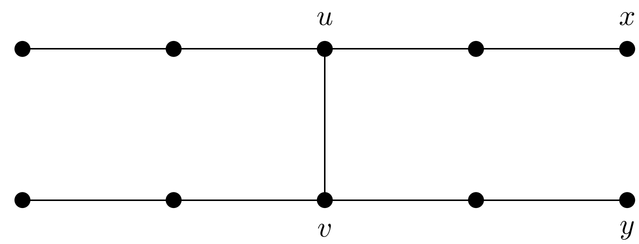

Figure 1: A tree T

but $\delta_R(P_{10}, T) = 2$ as one may rotate the edge uv to uy and thence to xy. The 'proof' in [8] that the value is exact assumes (incorrectly) that the diameter is a slowly-changing parameter under edge rotations. The following provides a suitable formula.

Theorem 2 *For all trees T of order n:* $\delta_R(P_n, T) = \text{end}(T) - \text{end}(P_n)$.

Proof

If $n = 1$ then the result is clearly true; therefore we assume that $n \geq 2$. We note first that for non-trivial trees T the parameters $\text{end}(T)$ and $\text{end}'(T)$ coincide. Thus we refer to $\text{end}(T)$ where, in fact, $\text{end}'(T)$ is the slowly-changing parameter (from Lemma 3). Further, the path is the only tree with exactly two end-vertices; thus we have verified properties **P1** and **P2** of Lemma 4. Property **P3** will follow when we show that for any tree T with more than two end-vertices there exists a tree T' formed by a single edge rotation but with one less end-vertex.

Let x be an end-vertex of T, and let y be the vertex of T, of degree at least three, nearest x. Let z be any neighbour of y *not* on the (x, y)-path. Then form T' by rotating zy to zx. As it is still connected, T' is a tree, and as x is no longer an end-vertex, $\text{end}(T') = \text{end}(T) - 1$. Thus the provisions of Lemma 4 are satisfied and the proof is complete. □

The result is not extendible to edge moves. Consider again the tree T of Figure 1. Clearly $\delta_M(P_{10}, T) = 1$ via the edge move uv to xy, while $\text{end}(T) - \text{end}(P_{10}) = 2$.

5 Bounds and Related Questions

We considered earlier some results linking edge moves, rotations and slides. The following result is an alternate formulation for the edge move distance.

Proposition 2 *Let G and H be graphs with order p and size q. Then $\delta_M(G, H) = q - s(G, H)$.*

Proof

We use Lemma 4 again, for we may let $s_H(G)$ denote $s(G, H)$ for fixed H and general G, so that the above statement claims that $\delta_M(G, H) = s_H(H) - s_H(G)$. We observed in Section 3 that $s_H(G)$ is slowly-changing with respect to edge moves, and clearly H is the only graph (of the prescribed order and size) with $s_H(H) = q$. We thus need to verify property **P3** of Lemma 4.

Let G be a graph of the prescribed order and size such that $s(G, H) < q$. Let F be a greatest common subgraph of G and H with vertex set $\{v_1, \dots, v_r\}$. Then label the vertices of G and H with $\{u_1, \dots, u_p\}$ and $\{w_1, \dots, w_p\}$, respectively, such that $F \cong F_G \leq \langle\{u_1, \dots, u_r\}\rangle$ and $F \cong F_H \leq \langle\{w_1, \dots, w_r\}\rangle$ via the natural bijections of their vertex sets. Now, let e be any edge in G but not in F_G and let e' be any edge in H but not in F_H. Then form G' by removing edge e and inserting edge e'. Clearly, $s_H(G') = s_H(G) + 1$ which implies **P3** and thus completes the provisions of Lemma 4. □

Thus the distance $\delta_M(G,H)$ is a special case of distances introduced by Baláž, Koča, Kvasníčka and Sekanina [1] and Johnson [4]. In [1] the distance between any two graphs G and H is defined by $q(G)+q(H)+|p(G)-p(H)|-2s(G,H)$, while in [4] it is defined as the minimum of $p(G)+p(H)+q(G)+q(H)-2(p(F)+q(F))$, taken over all F that are (isomorphic to) subgraphs of G and H. It is not difficult to see that these definitions represent the same distance measure. When G and H have the same order p and size q, this simplifies to $\delta(G,H) = 2q - 2s(G,H)$. Indeed, employing techniques similar to those used above, it may be shown that the above is the distance resulting from allowing, as permissible deformations, the addition or the deletion of an edge or isolated vertex.

As a simple consequence of Proposition 2 and Lemma 1 we may recover the following.

Corollary 3 [3] *Let G and H be graphs with order p and size q. Then $\delta_R(G,H) \leq 2(q - s(G,H))$.*

Chartrand et al. [3] also gave examples of equality in the corollary. Obtaining upper bounds for the edge slide distance is more problematic. As a consequence of Corollary 3 and Lemma 1, we may derive a result of the following form. If r is an upper bound on the diameter of the graphs $G, G_1, \ldots, G_{d-1}, H$ representing a minimum sequence of edge rotations from graph G to graph H, then $\delta_S(G,H) \leq r \cdot \delta_R(G,H) \leq 2r(q - s(G,H))$. However, a meaningful estimate of r is hard to come by, especially if G (say) is disconnected.

We now look at some related questions. In [3] it is shown that one can find graphs of the same order and size which are arbitrarily far apart with regard to edge rotation distance. Indeed, our Corollary 2 shows that for all

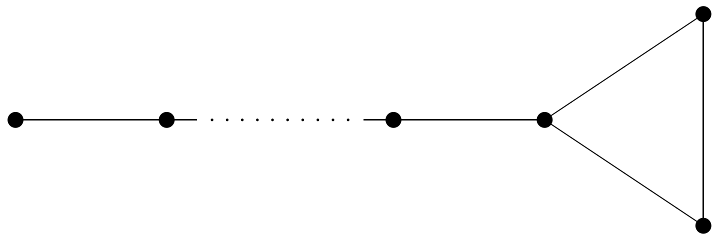

Figure 2: A connected unicyclic graph

three measures $\delta(S_{n+3}, P_{n+3}) = n$. One may extend this to obtain graphs arbitrarily far apart with a unique 'shortest path' between them. As an example, let A and B be two connected graphs which are distance one apart (e.g. P_4 and S_4). Then let $G = nA$ and $H = nB$. Clearly $\delta(G, H) = n$ and there is a unique shortest sequence of deformations from G to H (changing a copy of A to one of B at each step).

On the other hand, one may ask for graphs G and H which are such that $\delta_R(G, H) = 1$ while $\delta_S(G, H) = n$ for any prescribed positive integer n. As an example, let $G = C_{n+3}$ and for H take the (connected) unicyclic graph of order $n + 3$ in Figure 2. Clearly $\delta_R(G, H) = 1$ while Theorem 1 implies that $\delta_S(G, H) = n$.

In [3] it is noted that $\delta_R(G, H) = \delta_R(\bar{G}, \bar{H})$ as a rotation changing G to G' corresponds directly to a rotation changing $\bar{G}'$ to $\bar{G}$. While a similar result holds for the edge move distance, this duality does not carry over into edge slides. For example, by taking $G = \bar{C}_{n+3}$ and H to be the complement of the graph in Figure 2, one obtains graphs such that $\delta_S(G, H) = 1$ while $\delta_S(\bar{G}, \bar{H}) = n$. In general, one is not even guaranteed that $\delta_S(\bar{G}, \bar{H})$ is defined (see Proposition 1).

Finally, another question one may ask is: Can one obtain arbitrarily large sets of graphs which are mutually distance one apart? This is answered in the affirmative by the following construction.

Take a copy of K_n with vertex set $\{v_1, \ldots, v_n\}$. Then form H by attaching i 'feet' to each vertex v_i; i.e. introduce i new end-vertices adjacent to v_i only ($i = 1, 2, \ldots, n$). Then for each value of i, $i = 1, \ldots, n$, form the graph H_i by taking the graph H and introducing a single end-vertex x adjacent to v_i only. Clearly these H_i are all distinct and two distinct H_i are distance one apart (be it edge move, rotation or slide distance). By taking nH_i rather than just H_i one may obtain arbitrarily large sets of graphs mutually distance n apart.

The questions answered here are but a few of the possible questions. Nevertheless, they clearly show the interplay among the three measures, and between them and slowly-changing parameters.

Acknowledgement

The authors would like to thank Professor Gary Chartrand for introducing them to distance measures and to thank Western Michigan University for the use of its facilities. Further, the research of Terry McKee was supported in part by the U.S. Office of Naval Research contract N00014-88-K-0163.

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Authors' addresses:

Gerhard Benadé: Rand Afrikaans University, Johannesburg, 2000, South Africa

Wayne Goddard: Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Terry A. McKee: Wright State University, Dayton, Ohio 45435, USA

Paul A. Winter: University of Natal, Durban, 4000, South Africa

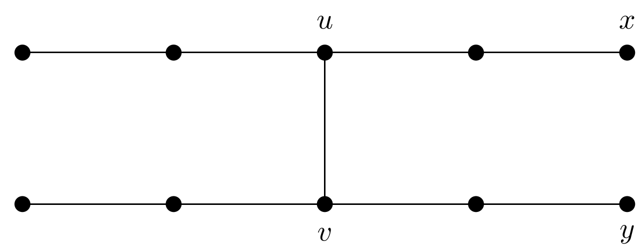

Figure 1: A tree T

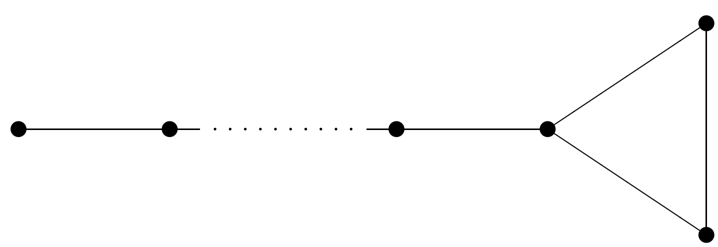

Figure 2: A connected unicyclic graph