The diameter of total domination vertex critical graphs

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Abstract

A graph G with no isolated vertex is total domination vertex critical if for any vertex v of G that is not adjacent to a vertex of degree one, the total domination number of $G - v$ is less than the total domination number of G. These graphs we call γ_t-critical. If such a graph G has total domination number k, we call it k-γ_t-critical. We characterize the connected graphs with minimum degree one that are γ_t-critical and we obtain sharp bounds on their maximum diameter. We calculate the maximum diameter of a k-γ_t-critical graph for $k \leq 8$ and provide an example which shows that the maximum diameter is in general at least $5k/3 - O(1)$.

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1 Introduction

For many graph parameters, criticality is a fundamental question. Much has been written about those graphs where a parameter (such as connectedness or chromatic number) goes up or down whenever an edge or vertex is removed or added. For domination number (the smallest cardinality of a set whose closed neighborhood is the whole graph), Brigham, Chinn, and Dutton [1] began the study of those graphs where the domination number decreases on the removal of any vertex. These we call γ-critical. Further properties of these graphs were explored in [3, 4, 8, 9, 10, 11], but they have not been characterized.

In this paper we introduce the same concept for total domination. We use the notation of [6]. In particular, if $G = (V, E)$ denotes a graph, then the (open) neighborhood of vertex $v \in V$ is denoted by $N(v) = \{u \in V \mid uv \in E\}$ while $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The set S is a *dominating set* if $N[S] = V$, and a *total dominating set* if $N(S) = V$. For sets $S, T \subseteq V$, S *totally dominates* T if $T \subseteq N(S)$. The minimum cardinality of a total dominating set is the *total domination number*, denoted $\gamma_t(G)$. A total dominating set of cardinality $\gamma_t(G)$ we call a $\gamma_t(G)$-set. For a detailed treatment of this parameter, the reader is referred to [6].

For a set $S \subseteq V$, we denote the subgraph of G induced by S by $G[S]$. The minimum and maximum degrees of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. An *end-vertex* is a vertex of degree one and a *support vertex* is one that is adjacent to an end-vertex. Let $S(G)$ be the set of support vertices of G. We say that a vertex $v \in V$ is *critical* if $\gamma_t(G - v) < \gamma_t(G)$. Since total domination is undefined for a graph with isolated vertices, we say that a graph G is *total domination vertex critical*, or just γ_t-critical, if every vertex of $V - S(G)$ is critical. If G is γ_t-critical, and $\gamma_t(G) = k$, then we say that G is *k-γ_t-critical*. For example, the 5-cycle is 3-γ_t-critical and the 6-cycle is 4-γ_t-critical.

Note that a graph is γ_t-critical if and only if each component is γ_t-critical. Also, K_2 is trivially 2-γ_t-critical. So henceforth we consider only connected graphs of order at least 3. The removal of a vertex can decrease the total domination number by at most one. Hence:

Observation 1 *If G is a γ_t-critical graph, then $\gamma_t(G - v) = \gamma_t(G) - 1$ for every $v \in V - S(G)$. Furthermore, a $\gamma_t(G - v)$-set contains no neighbor of v.*

Next we observe a sufficient condition for a graph not to be γ_t-critical.

Observation 2 *If a graph G has nonadjacent vertices u and v with $v \notin S(G)$ and with $N(u) \subseteq N(v)$, then G is not γ_t-critical.*

Proof. Let S be a $\gamma_t(G - v)$-set. In order for u to be totally dominated, there is a vertex $x \in N(u) \cap S$. Since $N(u) \subseteq N(v)$, S also totally dominates $V(G)$, and so $\gamma_t(G) \leq |S| = \gamma_t(G - v)$. Thus, G is not γ_t-critical. □

We proceed as follows. In Section 2 we characterize the connected γ_t-critical graphs that have an end-vertex, and we obtain sharp bounds on their maximum diameter. In Section 3 we consider the maximum diameter of a k-γ_t-critical graph. Finally, in Section 4 we briefly discuss links between k-γ_t-critical graphs and related families and list some open problems.

2 Graphs with End-Vertices

In this section we characterize the γ_t-critical graphs with end-vertices. For this purpose, we recall that the *corona* $\mathrm{cor}(H)$ of a graph H (denote $H \circ K_1$ in [6]) is that graph obtained from H by adding a pendant edge to each vertex of H.

Theorem 3 *Let G be a connected graph of order at least 3 with at least one end-vertex. Then, G is k-γ_t-critical if and only if $G = \mathrm{cor}(H)$ for some connected graph H of order k with $\delta(H) \geq 2$.*

Proof. Suppose $G = \mathrm{cor}(H)$ for some connected graph H of order k with $\delta(H) \geq 2$. Then $\gamma_t(G) = |V(H)| = k$. Let $u \in V(G) - S(G)$. Then $\deg u = 1$ and u is adjacent to a unique vertex v of H. Since $\delta(H - v) \geq 1$, it follows that $V(H) - \{v\}$ totally dominates $G - u$, and so $\gamma_t(G - u) \leq |V(H)| - 1 = \gamma_t(G) - 1$. Thus, G is k-γ_t-critical.

Now, suppose that G is a k-γ_t-critical graph with $\delta(G) = 1$. Let v' be an end-vertex and let v be its neighbor. Suppose there exists $w \in N(v) - \{v'\}$ with $w \notin S(G)$. Then by Observation 1, a $\gamma_t(G - w)$-set does not contain v, but v is required to totally dominate v', a contradiction. Thus each vertex in $N(v) - \{v'\}$ is a support vertex. It follows that $G = \mathrm{cor}(H)$ for some connected graph H of order $k \geq 2$. In particular, $\gamma_t(G) = |V(H)| = k$.

Suppose that H has an end-vertex v. Let w be the neighbor of v in H, and let w' be the end-vertex of G adjacent to w. Let S' be a $\gamma_t(G - w')$-set. Then, $V(H) - \{w\} \subset S'$. In order to totally dominate v, S' must contain v' or w, and so $|S'| \geq |V(H)| = \gamma_t(G)$, a contradiction. Hence, $\delta(H) \geq 2$. □

As a consequence of Theorem 3, we have the following corollaries.

Corollary 4 *No tree is γ_t-critical.*

Corollary 5 *If G is a connected k-γ_t-critical graph with at least one end-vertex, then $\mathrm{diam}(G) \leq k$ if $k \in \{3, 4\}$ and $\mathrm{diam}(G) \leq k - 1$ if $k \geq 5$, and these bounds are sharp.*

Proof. By Theorem 3, $G = \mathrm{cor}(H)$. Hence, $\mathrm{diam}(G) = 2 + \mathrm{diam}(H)$. If $k = 3$, then $H = K_3$ and so $\mathrm{diam}(G) = 3$. If $k = 4$, then C_4 is a subgraph of H and so $\mathrm{diam}(G) \leq 4$. For $k \geq 5$, it is a simply exercise to show that the maximum diameter of a graph H on k vertices with minimum degree 2 is $k - 3$. For $k \in \{3, 4, 5\}$, an extremal H is a cycle, while for $k \geq 6$, an extremal H is obtained from two disjoint triangles by joining them with an edge and then subdividing this edge $k - 6$ times. □

The above theorem is also useful in proving a characterization of 3-γ_t-critical graphs. A graph H is *vertex diameter k-critical* if $\mathrm{diam}(H) = k$ and $\mathrm{diam}(H - v) > k$ for all $v \in V(H)$. Hanson and Wang [5] observed the following result.

Theorem 6 [5] *For a graph G, $\gamma_t(G) = 2$ if and only if the complement $\overline{G}$ has diameter greater than two.*

Theorem 7 *A connected graph G is 3-γ_t-critical if and only if $\overline{G}$ is vertex diameter 2-critical or G is the net,* $\mathrm{cor}(K_3)$.

Proof. Assume G is 3-γ_t-critical. Since $\gamma_t(G) > 2$, Theorem 6 implies that $\mathrm{diam}(\overline{G}) = 2$. If G has a pendant edge, then by Theorem 3, G is the net. If G has no pendant edge, then $\gamma_t(G - v) = 2$ for all $v \in V$, and so, by Theorem 6, $\mathrm{diam}(\overline{G - v}) > 2$. Thus, $\overline{G}$ is vertex diameter 2-critical. Conversely, if $\overline{G}$ is vertex diameter 2-critical, then Theorem 6 implies that $\gamma_t(G) \geq 3$ and $\gamma_t(G - v) = 2$. Since G is connected, $\gamma_t(G) \leq 1 + \gamma_t(G - v)$, and so G is 3-γ_t-critical. □

For example, the Petersen graph is vertex diameter 2-critical and so the complement is 3-γ_t-critical.

3 Bounds on the Diameter

In this section we establish bounds on the diameter of a connected k-γ_t-critical graph. We first determine which cycles are γ_t-critical. To this end we recall the total domination numbers of path P_n and cycle C_n on n vertices.

Observation 8 [7] *For $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.*

Proposition 9 *A cycle C_n is γ_t-critical if and only if $n \equiv 1, 2 \pmod 4$.*

Proposition 9 shows that the diameter of a k-γ_t-critical graph can be linear in k. We provide next a trivial upper bound on the diameter of a k-γ_t-critical graph G. Throughout this section for $x \in V$, we let S_x denote a $\gamma_t(G - x)$-set.

Proposition 10 *The diameter of a k-γ_t-critical graph G is at most $2k - 3$.*

Proof. Let v be a diametrical vertex of G. Let $d = \mathrm{diam}(G)$. For $i = 0, 1, \ldots, d$, let V_i denote the set of all vertices of G at distance i from v. In particular, $V_0 = \{v\}$ and $V_1 = N(v)$. Then, $|S_v| = k - 1$. By Observation 1, $S_v \cap V_1 = \emptyset$. Hence to totally dominate V_1, $|S_v \cap V_2| \geq 1$. In fact by Observation 2, $|S_v \cap V_2| \geq 2$. Thus, $S = S_v \cup \{v_1\}$ is a $\gamma_t(G)$-set for any $v_1 \in V_1$ and $|S \cap (V_1 \cup V_2)| \geq 3$. For any $i \geq 3$, $|S \cap (V_i \cup \cdots \cup V_{i+3})| \geq 2$. It follows that if $d = 2 + 4j + r$ where $0 \leq r \leq 3$, then $k = |S| \geq 3 + 2j$ if $r \in \{0, 1\}$ while $k \geq 3 + 2j + \lceil r/2 \rceil$ if $r \in \{2, 3\}$. In all cases, $d \leq 2k - 3$ with inequality if $d \not\equiv 3 \pmod 4$. □

Next we establish a sharp upper bound on the diameter of a connected k-γ_t-critical graph for small k.

Theorem 11 *For $k \leq 8$, the diameter of a k-γ_t-critical graph is at most the value given by the following table:*

k	3	4	5	6	7	8
diam	3	4	6	7	9	11

Proof. If $\delta(G) = 1$, then the upper bounds follow from Corollary 5. If $k = 3$, then the upper bound follows from Proposition 10. Hence we may assume $\delta(G) \geq 2$ and $k \geq 4$. Let v be a diametrical vertex of G. Let $d = \mathrm{diam}(G)$. For $i = 0, 1, \ldots, d$, let V_i denote the set of all vertices of G at distance i from v. In particular, $V_0 = \{v\}$ and $V_1 = N(v)$. For $i \in \{0, 1, \ldots, d\}$, let

$$V_{\leq i} = \bigcup_{j=0}^{i} V_j \quad \text{and} \quad V_{\geq i} = \bigcup_{j=i}^{d} V_j.$$

Let $V = V(G)$. For subsets S and T of V, we write $S \succ_t T$ if S totally dominates T in G. Furthermore, we write $S \mapsto_t T$ if $S \cap T \succ_t T$. As before, for $u \in V$, let S_u be a $\gamma_t(G - u)$-set.

Case 1: $k = 4$.

Let $u \in V_1$. Then $|S_u| = 3$. To totally dominate V_0, $|S_u \cap V_1| \geq 1$. Since $G[S_u]$ must be connected, it follows that $S_u \subset V_{\leq 3}$, and so $d \leq 4$.

Case 2: $k = 5$.

Suppose $d \geq 7$. Let $u \in V_1$; then $|S_u| = 4$. To totally dominate $V_0 \cup V_4 \cup V_7$, it follows that $d = 7$ and $|S_u \cap V_j| = 1$ for $j \in \{1, 2, 5, 6\}$. Then $S_u \mapsto_t V_{\geq 4}$. By symmetry, for $w \in V_6$ it follows that $|S_w \cap V_{\leq 3}| = 2$ and $S_w \mapsto_t V_{\leq 3}$. Therefore, $(S_u \cap V_{\geq 4}) \cup (S_w \cap V_{\leq 3}) \succ_t V$, which contradicts $\gamma_t(G)$ being 5.

Case 3: $k = 6$.

Suppose $d \geq 8$. Let $u \in V_1$; then $|S_u| = 5$. To totally dominate $V_0 \cup V_4 \cup V_5 \cup V_8$, it follows that $d = 8$, and $|S_u \cap V_j| = 1$ for either $j \in \{1, 2, 3, 6, 7\}$ or $j \in \{1, 2, 5, 6, 7\}$.

Let $w \in V_7$. As before, $|S_w \cap V_j| = 1$ for either $j \in \{1, 2, 5, 6, 7\}$ or $j \in \{1, 2, 3, 6, 7\}$. Then $S_w \mapsto_t V_{\leq 3}$ while $S_u \cap V_{\geq 3} \succ_t V_{\geq 4}$. So, if $S_w \cap V_3 = \emptyset$, then $\gamma_t(G) \leq 5$, a contradiction. Hence we may assume that $S_w \cap V_3$ is nonempty; similarly we may assume that $S_u \cap V_5$ is nonempty.

Let $x \in V_4$. To totally dominate V_0 and V_8, it follows that $|S_x \cap V_{\leq 2}|, |S_x \cap V_{\geq 6}| \geq 2$. Since no vertex of $G[S_x]$ is isolated, $S_x \cap V_4 = \emptyset$. Without loss of generality we may assume that $|S_x \cap V_{\leq 3}| = 2$. Then, $S_x \mapsto_t V_{\leq 3}$ while $S_u \mapsto_t V_{\geq 4}$; so $\gamma_t(G) \leq 5$, a contradiction.

Case 4: $k = 7$.

Suppose $d \geq 10$. Let $u \in V_1$; then $|S_u| = 6$. As before, $|S_u \cap V_{\leq 2}|, |S_u \cap V_{\geq 8}| \geq 2$. Hence, to totally dominate $V_4 \cup V_5 \cup V_6$, $|S_u \cap V_5| \geq 1$ and $|S_u \cap (V_4 \cup V_5 \cup V_6)| \geq 2$. Hence, $|S_u \cap V_{\leq 2}| = 2$, $|S_u \cap V_{\geq 8}| = 2$, $|S_u \cap V_5| \geq 1$, and $|S_u \cap (V_4 \cup V_5 \cup V_6)| = 2$. In particular, $S_u \mapsto_t V_{\geq 4}$.

Let $w \in V_9$. By symmetry, $|S_w \cap V_{\leq 2}| = 2$, $|S_w \cap V_{\geq 8}| = 2$, $|S_w \cap V_5| \geq 1$, and $|S_w \cap (V_4 \cup V_5 \cup V_6)| = 2$. In particular, $S_w \mapsto_t V_{\leq 6}$. If $S_u \cap V_6 = \emptyset$, then $S_u \mapsto_t V_{\geq 7}$, and

$(S_w \cap V_{\leq 6}) \cup (S_u \cap V_{\geq 8}) \succ_t V$ which contradicts $\gamma_t(G)$ being 7. Thus we may assume that $|S_u \cap V_6| = 1$, and so $|S_u \cap V_5| = 1$; similarly, $|S_w \cap V_4| = |S_w \cap V_5| = 1$.

Let $x \in V_5$. Then, as before, $|S_x \cap V_{\leq 2}| = 2$ and $|S_x \cap V_{\geq 8}| = 2$. Suppose there is another vertex in V_5. Then $|S_x \cap V_5| \geq 1$, and $|S_x \cap (V_4 \cup V_5 \cup V_6)| = 2$. Without loss of generality, $S_x \cap V_4 = \emptyset$. So, $S_x \mapsto_t V_{\leq 3}$. Therefore $(S_x \cap V_{\leq 2}) \cup (S_u \cap V_{\geq 4}) \succ_t V$ which contradicts $\gamma_t(G)$ being 7. Hence there is no other vertex in V_5, and so $|S_x \cap V_j| = 1$ for $j \in \{1, 2, 3, 7, 8, 9\}$ and $d = 10$.

Let $y \in V_6$. By Observation 1, since x dominates V_6, $x \notin S_y$. As before, $|S_y \cap V_{\leq 2}|, |S_y \cap V_{\geq 8}| \geq 2$. If $|S_y \cap V_4| = 0$, then to totally dominate $V_{\leq 4}$ it follows that (since $x \notin S_y$) $|S_y \cap V_{\leq 3}| \geq 3$, while to totally dominate $\{x\} \cup V_{10}$ it follows that $|S_y \cap V_{\geq 6}| \geq 4$, and so $|S_y| \geq 7$, a contradiction. Hence, $|S_y \cap V_4| \geq 1$ and thus $|S_y \cap V_{\leq 4}| \geq 4$. In particular, this implies that $V_6 = \{y\}$ and that the two vertices of $S_y \cap V_{\geq 8}$ totally dominate $V_{\geq 7}$. But then $(S_w \cap V_{\leq 6}) \cup (S_y \cap V_{\geq 8})$ is a total dominating set of G of cardinality 6, a contradiction.

Case 5: $k = 8$**.**
Suppose $d \geq 12$. Let $u \in V_1$; then $|S_u| = 7$. As before, $|S_u \cap V_{\leq 2}|, |S_u \cap V_{\geq 10}| \geq 2$. Hence $|S_u \cap V_j| = 1$ for all j in either $\{1, 2, 5, 6, 7, 10, 11\}$, $\{1, 2, 5, 6, 9, 10, 11\}$, or $\{1, 2, 3, 6, 7, 10, 11\}$. So $S_u \cap V_{\geq 3} \succ_t V_{\geq 4}$.

Let $w \in V_{11}$. Then, by symmetry, $|S_w \cap V_j| = 1$ for all j in one of the three possibilities given above. Then $(S_w \cap V_{\leq 3}) \cup (S_u \cap V_{\geq 3}) \succ_t V$, and so contradicts $\gamma_t(G)$ being 8 unless $|S_w \cap V_{\leq 3}| = 3$. Thus we may assume that $|S_w \cap V_j| = 1$ for all j in $\{1, 2, 3, 6, 7, 10, 11\}$. By symmetry, $|S_u \cap V_j| = 1$ for all j in $\{1, 2, 5, 6, 9, 10, 11\}$. In particular, $S_w \mapsto_t V_{\leq 8}$ and $S_u \mapsto_t V_{\geq 4}$.

Now let $y \in V_4$. To totally dominate V_6 it follows that $|S_y \cap (V_5 \cup V_6 \cup V_7)| \geq 2$. Hence either $|S_y \cap V_{\leq 4}| = 2$ or $|S_y \cap V_{\geq 8}| = 2$. In the former case, let $S = (S_y \cap V_{\leq 2}) \cup (S_u \cap V_{\geq 5})$, while in the latter case, let $S = (S_w \cap V_{\leq 7}) \cup (S_y \cap V_{\geq 10})$. In both cases, S is a total dominating set of G of cardinality 7, a contradiction. □

3.1 Constructions

First we give a way of constructing a critical graph from two smaller critical graphs.

Lemma 12 *Let F and H be j-γ_t-critical and k-γ_t-critical graphs, respectively, with minimum degrees at least two and let G be a graph formed by identifying a vertex of F with a vertex of H. If $\gamma_t(G) = j + k - 1$, then G is γ_t-critical.*

Proof. Note that since $\delta(F) \geq 2$ and $\delta(H) \geq 2$, $S(G) = \emptyset$. Label the identified vertex v. Let $u \in V(G)$. Without loss of generality, $u \in V(F)$. Since F is j-γ_t-critical, $\gamma_t(F - u) = j - 1$. If $u \neq v$, then any $\gamma_t(F - u)$-set dominates v implying that only $\gamma_t(H - v) = k - 1$ vertices are needed to totally dominate H. Hence, $\gamma_t(G - u) \leq j - 1 + k - 1 < \gamma_t(G)$. If $u = v$, then $\gamma_t(G - v) = \gamma_t(F - v) + \gamma_t(H - v) = j - 1 + k - 1 < \gamma_t(G)$. Thus, $\gamma_t(G - u) < \gamma_t(G)$ and G is γ_t-critical. □

We define a graph as *pointed* if there are two designated diametrical vertices called LEFT and RIGHT. Then for two pointed graphs G and H, we define $G \circ H$ as the pointed graph obtained by identifying and undesignating the RIGHT-vertex from G and the LEFT-vertex from H. Note that the operator $\circ$ is associative.

Now we define our building blocks. Let H_1 be a copy of P_4 and let H_2 be a copy of $\overline{H_1}$. Let F be the pointed graph obtained from $H_1 \cup H_2$ by adding all edges between H_1 and H_2 except for a perfect matching between corresponding vertices of H_1 and H_2, and then adding two new vertices LEFT and RIGHT such that LEFT is joined to every vertex in H_1 and RIGHT is joined to every vertex in H_2. The graph is shown in Figure 1 where for clarity we omit the edges between H_1 and H_2. It is straightforward to check that F is 3-γ_t-critical with diameter 3.

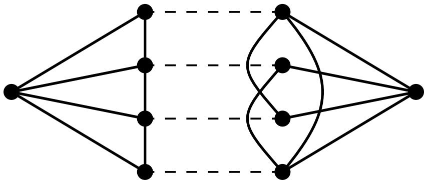

Figure 1: The 3-γ_t-critical graph F of diameter 3

Let R be the pointed graph on 17 vertices defined as follows. Let $S = \{s_1, s_2, s_3, s_4\}$, $T = \{t_1, t_2, t_3, t_4\}$ and $U = \{u_1, u_2, u_3, u_4\}$. Add edges such that s_1, s_2, s_3, s_4, s_1 induces a cycle, and t_3, t_1, t_4, t_2 and u_3, u_1, u_4, u_2 induce P_4s. Add all edges between S and T except for a perfect matching between corresponding vertices; similarly with S and U. Add all edges between T and U. Add two new vertices a and a' such that a is adjacent to $\{s_2, s_3, s_4, t_2, t_3, u_2, u_3\}$ and a' is adjacent to $\{s_1, s_2, s_3, t_2, t_3, u_2, u_3\}$. Finally add three new vertices, LEFT, r', and RIGHT, such that LEFT is adjacent to all of S, RIGHT is adjacent to all of $T \cup U \cup \{a\}$, and r' is adjacent to all of $T \cup U \cup \{a'\}$. It is straightforward, though tedious, to check that R is 3-γ_t-critical with diameter 3.

The graph R has more properties than F, and so in later discussion we can replace F by R. However, wherever possible we use F because it is simpler.

Let J_1 and J_3 be disjoint copies of $2K_2$ and let J_2 be a copy of $\overline{J_1}$. Let J be the pointed graph obtained from $J_1 \cup J_2 \cup J_3$ by adding all edges between J_1 and J_2 (respectively, J_2 and J_3) except for a perfect matching between corresponding vertices of J_1 and J_2 (respectively, J_2 and J_3), and then adding two new vertices LEFT and RIGHT such that LEFT is adjacent to all of J_1 and RIGHT is adjacent to all of J_3. It is straightforward to check that J is 4-γ_t-critical with diameter 4.

Finally, let Q be the pointed graph obtained from $F \circ F$ by deleting the cut-vertex v and adding all edges joining the four neighbors of v in the one copy of F to the four neighbors of v in the other copy. See Figure 2. It is straightforward to check that $\gamma_t(Q) = 4$ and Q has diameter 5. (The graph Q is not 4-γ_t-critical.)

Observe that each of the pointed graphs F, R, J, and Q has a γ_t-set containing the vertex LEFT and a γ_t-set containing the vertex RIGHT. With these constructions, one can show

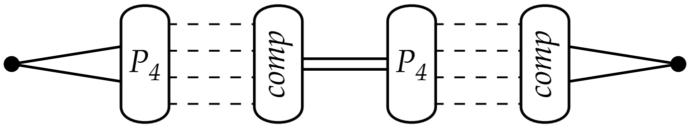

Figure 2: The pointed graph Q

that the bounds in Theorem 11 are best possible. Examples are given in the following table, where the last line of the table (when $k = 8$) will be proved in Theorem 13.

k	graph with max diam
3	F
4	J
5	$F \circ F$
6	$F \circ J$
7	$F \circ R \circ F$
8	$F \circ Q \circ F$

In general we have the following.

Theorem 13 *For all* $k \equiv 2 \pmod 3$, *there exists a* k-γ_t-*critical graph of diameter* $(5k - 7)/3$.

Proof. For $q \geq 0$, define the pointed graph $Y_q = F \circ Q \circ \cdots \circ Q$ for q copies of Q and define $Z_q = Y_q \circ F$. Then $\mathrm{diam}(Z_q) = 5q + 6$. For a pointed graph G, we define $L_i(G)$ as the vertices at distance i from LEFT and $R_i(G)$ as the vertices at distance i from RIGHT.

Claim 1 *(a)* $\gamma_t(Y_q) \geq 3q + 3$; *if* T *totally dominates* $Y_q - \{\text{RIGHT}\}$, *then* $|T| \geq 3q + 2$.
(b) $\gamma_t(Z_q) \geq 3q + 5$.
(c) $\gamma_t(Z_q - v) \leq 3q + 4$ *for all* $v \in V(Z_q)$.

Proof. (a) By induction on q. Note that $Y_0 = F$; so the base case is true. Assume then $q \geq 1$. Label the i^{th} copy of Q by Q_i. Note that $Y_q = Y_{q-1} \circ Q_q$, and let x be the vertex so identified.

Let S be a total dominating set of Y_q. Define S' as the intersection of S with $V(Q_q) - \{x\}$. Then it can be checked that $|S'| \geq 3$. Furthermore, if $|S'| = 3$, then S' does not dominate x; and so $|S - S'| \geq \gamma_t(Y_{q-1}) \geq 3q$. On the other hand, in general $S - S'$ totally dominates $Y_{q-1} - \{x\}$, and so has at least $3q - 1$ elements. In either case, $|S| \geq 3q + 3$.

Let T be a total dominating set of $Y_q - \{\text{RIGHT}\}$ and define T' as the intersection of T with $V(Q_q) - \{x\}$. By similar arguments, $|T'| \geq 2$, and if equality holds, then $T - T'$ totally dominates Y_{q-1}. Thus, $|T| \geq 3q + 2$.

(b) Note that $Z_q = Y_{q-1} \circ F$. Then using the same approach as in (a), except with S' defined as the intersection of S with the non-cut-vertices of the final copy of F, the desired result follows readily.

(c) Assume v was from a copy of F (without loss of generality, the left one). Construct a set S_v as follows. Since F is 3-γ_t-critical, there exist two vertices which totally dominate $F - v$. Also, there exist two vertices in $L_2(Q_1)$ which totally dominate $L_1(Q_1) \cup L_2(Q_1) \cup L_3(Q_1)$. Thereafter take one vertex from each of the first three levels of each block; that is, a vertex from each of $L_0(B)$, $L_1(B)$ and $L_2(B)$ for $B = Q_2, \ldots, Q_q, F$. These can be chosen such that S_v dominates $Z_q - v$. The set has cardinality $3q + 4$.

Assume v was from a copy of Q; by symmetry, without loss of generality, $v \in L_0(Q_i) \cup L_1(Q_i) \cup L_2(Q_i)$. Construct a set S_v as follows. Since F is 3-γ_t-critical, there exist two vertices in $L_1(Q_i) \cup L_2(Q_i)$ which totally dominate $L_0(Q_i) \cup L_1(Q_i) \cup L_2(Q_i) \cup L_3(Q_i)$ except for v. To the right, take triples at the start of blocks as before (that is, a vertex from each of $L_0(B)$, $L_1(B)$ and $L_2(B)$ for $B = Q_{i+1}, \ldots, Q_q, F$). To the left, take two vertices from $R_2(Q_{i-1})$ (or $R_2(F)$ if $q = 1$), and thereafter triples as before (that is, a vertex from each of $R_0(B)$, $R_1(B)$ and $R_2(B)$ for $B = Q_{i-2}, \ldots, Q_1, F$). These can be chosen such that S_v dominates $Z_q - v$. The set has cardinality $3q + 4$. This completes the proof of the claim. □

By Claim 1, Z_q is $(3q + 5)$-γ_t-critical. For $k = 2$ the desired graph is K_2. For $k \geq 5$, the graphs $Z_{(k-5)/3}$ have the desired properties. □

4 Open Questions

We close with a list of open problems and questions.

1. Characterize the 3-γ_t-critical graphs with diameter 3. Does there exist a 4-γ_t-critical graph with diameter 2?
2. Consider the connection between γ-critical and γ_t-critical graphs. For example, $K_3 \times K_3$ is γ-critical but not γ_t-critical. The cycle C_5 is γ_t-critical, but not γ-critical. So, which graphs are vertex domination critical and total domination vertex critical (or one but not the other)?
3. Determine the maximum diameter of a k-γ_t-critical graph.
4. If G is a γ_t-critical graph of order n, then it can be shown that $n \leq \Delta(G)(\gamma_t(G) - 1) + 1$. Characterize those graphs achieving equality.
5. Cockayne et al. [2] showed that if G is a connected graph of order $n \geq 2$, then $\gamma_t(G) \leq \max(n - \Delta(G), 2)$. Characterize γ_t-critical graphs G with $\gamma_t(G) = n - \Delta(G)$.

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