

The Toughness of Cubic Graphs

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Abstract

The toughness of a graph G is the minimum of $|S|/k(G - S)$ over all sets S of vertices such that $k(G - S) \geq 2$. In this paper upper bounds on the toughness of a cubic graph are derived in terms of the independence number and coloring parameters. These are applied to cycle permutation graphs.

Running head: as per title.

1 Introduction

For a graph G let $k(G)$ denote the number of components of G . Then the *toughness* $\tau(G)$ of G is the minimum of $|S|/k(G - S)$ taken over all sets S of vertices such that $k(G - S) \geq 2$. A set S for which the minimum is achieved is called a *tough* set.

The parameter toughness was introduced by Chvátal [3]. Though much of the research has focussed on the relationship between toughness and hamiltonicity, some general results were derived by Pippert [8] and Goddard and Swart [5].

We will denote the (vertex) connectivity of a graph G by $\kappa(G)$, the maximum degree by $\Delta(G)$, and the maximum cardinality of an independent set by $\beta(G)$. The order of G is the number of vertices it has.

A general bound is:

Proposition 1 [3, 5] *For a noncomplete graph G*

$$\frac{\kappa(G)}{\Delta(G)} \leq \tau(G) \leq \frac{\kappa(G)}{2}.$$

In his original paper Chvátal looked briefly at the toughness of regular graphs. We define a (noncomplete) r -regular graph as *supertough* if its toughness is $r/2$. For 3-regular graphs Chvátal showed that:

Proposition 2 [3] *If a noncomplete cubic graph is supertough then its order must be a multiple of 3.*

He conjectured that this behavior was typical of r -regular graphs when r is odd. This is not true. For, Matthews and Sumner [6] determined the toughness of graphs which do not have an induced copy of the claw $K(1, 3)$:

Proposition 3 [6] *If G is a noncomplete $K(1, 3)$ -free graph G then $\tau(G) = \kappa(G)/2$.*

For $r \geq 4$ an r -regular r -connected $K(1, 3)$ -free graph with large order need not have order a multiple of r . (For example, the line graph of a random $(3, 4)$ -biregular graph is 5-regular and almost surely 5-connected.) In fact, we believe that for any r sufficiently large it holds that almost every r -regular graph is supertough.

In this paper we derive upper bounds on the toughness of cubic graphs in terms of the independence number and coloring parameters. This enables us to establish a conjecture of Piazza, Ringeisen and Stueckle [7] about the toughness of cycle permutation graphs.

2 Upper Bounds

In this section we extend Proposition 2 to obtain upper bounds for the toughness of cubic graphs.

By a *coloring* of a graph G we mean an assignment of colors to the vertices of G such that any two vertices joined by an edge receive different colors (i.e., all our colorings are “proper”). A color class A is *minimal* if every vertex of A has a neighbor of every other color.

Lemma 1 *Let G be a cubic graph with a 3-coloring (A, B, C) such that A is minimal. If $|A| = a$, $|B| = b$ and $|C| = c$ then*

$$\tau(G) \leq \frac{3b + a - c}{2b}.$$

PROOF. Define A_C as the set of vertices of A with exactly one neighbor in C , and $A_B = A - A_C$. Let $S = A_C \cup C$ and consider its removal. In the remaining graph $G - S$ the vertices of A_B each have degree 1. Hence every vertex of B is in a separate component of $G - S$; so $k(G - S) = b$. Thus

$$\tau \leq (|A_C| + c)/b,$$

provided $b \geq 2$. If $b = 1$ then by the minimality of A it follows that $a \leq 3$ and that G has 6 vertices. The bound is easily checked in this case.

But we can calculate $|A_C|$! Let m_{AB} denote the number of edges between A and B , and define m_{AC} and m_{BC} analogously. Then $m_{AB} + m_{AC} = 3a$, $m_{AC} + m_{BC} = 3c$ and $m_{AB} + m_{BC} = 3b$, so that $m_{AB} = 3(a + b - c)/2$.

Since A is minimal it holds that $m_{AB} = 2|A_C| + |A_B|$. Thus $|A_C| = (3b + a - 3c)/2$. By plugging the value of $|A_C|$ into the above bound on τ the result follows. QED

The calculation of $|A_C|$ also shows that for any 3-coloring with minimal A :

$$3b + a \geq 3c. \tag{1}$$

(This fact can also easily be derived directly.)

Our first theorem gives an upper bound on toughness in terms of the independence number.

Theorem 1 *For a noncomplete cubic graph G on n vertices and independence number β :*

$$\tau(G) \leq \min\left(\frac{2n - 3\beta}{n - \beta}, \frac{2\beta}{4\beta - n}\right).$$

PROOF. By a theorem of Catlin [1] a graph of maximum degree r has an r -coloring where one of the color classes is a maximum independent set. Let (A, B, C) be a 3-coloring of G where C is a maximum independent set, and, subject to this, A is as small as possible. So $|C| = \beta$ and $b \geq (n - \beta)/2$. Also $3b \geq 3c - a = 3\beta - (n - b - \beta)$ by Inequality 1, whence $b \geq 2\beta - n/2$.

So by the above lemma,

$$\tau \leq \frac{3b + a - c}{2b} = 1 + \frac{n - 2\beta}{2b} \leq \min\left(\frac{2n - 3\beta}{n - \beta}, \frac{2\beta}{4\beta - n}\right),$$

as required. QED

This is best possible. For $\beta = n/2$ the theorem gives an upper bound of 1, and any 3-connected cubic bipartite graph has toughness 1. At the other extreme the theorem shows that $\tau = 3/2$ in noncomplete cubic graphs requires that $\beta = n/3$ and thus n a multiple of 3, which recovers Proposition 2.

Consider also the following cubic graph G_m for m a positive integer. Start with a set $U = \{u_1, \dots, u_{6m}\}$ of $6m$ vertices which form a cycle. Then add a set $V = \{v_1, \dots, v_{3m}\}$ of $3m$ vertices, and connect v_i to u_{2i-1} and u_{2i} . Finally add a set $W = \{w_1, \dots, w_m\}$ of m vertices, and join w_i to v_{3i-2} , v_{3i} and v_{3i+2} (subscripts modulo $3m$). The graph is illustrated in Figure 1. The graph G_m has $n = 10m$ vertices, toughness $4/3$ and independence number $\beta = 2n/5$ (which is where the two bounds of Theorem 1 meet).

Figure 1 here

Lemma 2 *For $m \geq 1$ G_m has toughness $4/3$.*

PROOF. Let S be a tough set and suppose $G_m - S$ has k components. Note that by Proposition 1 $k \leq |S|$ (since G_m is 3-connected). Thus as $(|S| + 1)/(k + 1) \leq |S|/k$ we may assume that $G_m - S$ has no cut-vertex.

We show first that we may assume that $S \cap V = \emptyset$. Suppose for some w_i some of its neighbors are in S . Then in $G_m - S$ reinsert the neighbors of w_i and remove w_i instead. This action cannot join two components (since v_j 's two neighbors in U are already adjacent). The only way this action can reduce the number of components is if w_i was a singleton component in $G_m - S$. But that means all of w_i 's neighbors are in S and since $(|S| - 2)/(k - 1) < |S|/k$ we are better off, a contradiction. So this action does not decrease the number of components and does not increase the number of vertices removed. Hence we may assume that $S \cap V = \emptyset$.

Let w denote $|S \cap W|$. Since $G_m - S$ has no cut-vertex, for any w_i not in S the vertices $u_{6i-4}, u_{6i-3}, \dots, u_{6i+3}$ must all lie in the component with w_i . Denote the subpath $u_{6i-4}u_{6i-3} \dots u_{6i+3}$ by P_i . Now it is not hard to see that the best strategy, once $S \cap W$ is determined, is to remove every alternate vertex of U that lies outside the P_i corresponding to the $w_i \notin S$. The number u of vertices of U removed is equal to the number of components that remain. Also $u \leq 3w$. Hence $\tau \geq (w + 3w)/(3w) = 4/3$. QED

Another bound is the following:

Theorem 2 *Let G be a noncomplete cubic graph on n vertices. Let a denote the minimum number of vertices whose removal from G leaves a bipartite graph. Then*

$$\tau(G) \leq \frac{n}{n - a}.$$

And, in fact, $\tau \leq (n - 2)/(n - a - 1)$ if $n - a$ is odd.

PROOF. Consider a 3-coloring (A, B, C) of G with a color class A of size a . A is minimal. By the above lemma $\tau \leq (3b + a - c)/(2b)$ and $\tau \leq (3c + a - b)/(2c)$. So the toughness is at most the harmonic mean of the two bounds viz. $n/(n - a)$.

If $n - a$ is odd then one of b or c , say b , is at most $(n - a - 1)/2$. Since for fixed a the bound of the lemma is decreasing in b , we may assume that $b = (n - a - 1)/2$ and $c = (n - a + 1)/2$ so that the desired upper bound follows. QED

3 Cycle Permutation Graphs

A special class of cubic graphs are the *cycle permutation graphs*. A cycle permutation graph on $2m$ vertices is defined by taking two vertex-disjoint cycles on m vertices and adding a matching between the vertices of the two cycles.

In [7] Piazza, Ringelsen and Stueckle conjectured that the toughness of such a graph is at most $4/3$. This we almost establish:

Theorem 3 *Let G be a cycle permutation graph on $2m$ vertices. Then*

$$\tau(G) \begin{cases} \leq 4/3 & m \equiv 0, 1 \pmod{4}, \\ < 4/3 & m \equiv 2 \pmod{4}, \\ \leq 4/3 + 4/(9m - 3) & m \equiv 3 \pmod{4}. \end{cases}$$

PROOF. Consider any 3-coloring \mathcal{C} of the first cycle of G . We claim that \mathcal{C} can be extended to a 3-coloring of all of G . To see this, let the uncolored cycle be given by $v_1v_2 \dots v_mv_1$, and suppose that v_1 's neighbor in the colored cycle, call it w_1 , has a different color to v_2 's neighbor in the colored cycle. Then color v_2 with w_1 's color. Thereafter color vertices v_3, v_4, \dots in turn greedily: each has only two neighbors already colored when we color it. Finally we get to v_1 . All of its neighbors have been colored, but v_2 and w_1 received the same color; so there is one color left over for v_1 . (This, of course, is the same idea used in the proof of Brooks.)

Now, to prove the corollary we start with a coloring of the first cycle that uses the first color, say A , as few times as possible: zero or one vertex receiving A depending on the parity of m . This extends to a three-coloring of G ; in that coloring A is used at most $\lceil m/2 \rceil$ times. Hence by Theorem 2 it follows that:

$$\tau \leq \frac{2m}{2m - \lceil m/2 \rceil}.$$

This establishes the above bound for $m \equiv 0$ or $3 \pmod{4}$. If $m \equiv 1$ or $2 \pmod{4}$ then $2m - \lceil m/2 \rceil$ is odd. So either $|A| \leq 2m - \lceil m/2 \rceil - 1$, or we can use the sharper bound in Theorem 2. The latter is the weaker bound, and gives the above values. QED

It is possibly interesting to note that the above proof can be recast. For example suppose m is even. Let S be a set consisting of a maximum independent set from both cycles chosen so that the minimum number of edges remain after the removal of S . To bound this minimum, consider taking the two maximum independent sets at random. The expected number of edges that remain is $m/4$ and thus the expected number of components after removal of the m vertices is $m - m/4$. Hence $\tau \leq m/(m - m/4) = 4/3$.

Information can also be extracted from the proof to provide some necessary conditions for equality in the bounds. However, it remains open whether there exists a family of cycle permutation graphs with toughness (asymptotically) $4/3$. Piazza, Ringelsen and Stueckle [7] suggested a specific subfamily of the ‘‘generalized Petersen graphs’’. But Ferland [4] showed that they have toughness about $5/4$. (This subfamily was also looked at by Chao and Han [2].) General lower bounds for the toughness of cycle permutation (and generalized Petersen) graphs remain elusive.

4 Supertough Cubic Graphs

Proposition 2 guarantees that a supertough cubic graph has order a multiple of 3. Proposition 3 shows that a 3-connected $K(1,3)$ -free cubic graphs is supertough. So:

Question. *Are all supertough cubic graphs $K(1,3)$ -free?*

What do we know about supertough cubic graphs? We call a cubic graph *balanced* if $\beta = n/3$. By Theorem 1 a supertough cubic graph is balanced. By Catlin's theorem being balanced is equivalent to every 3-coloring using each color $n/3$ times. It might be interesting to try to characterize balanced cubic graphs. It is known that such graphs must have a triangle (see for example [9]). But how many triangles? It can also be shown that they have no induced 5-cycle. Moreover a supertough cubic graph also cannot have a 4-cycle.

It is perhaps surprising that claw-free graphs are so tough. By extending the proof of Proposition 3 in [6] it can be shown that a 3-connected cubic graph has toughness at least $4/3$ if only one vertex is a center of a claw (equality for G_1 described above), and at least $5/4$ if two vertices are centers of claws. But nothing can be said about a 3-connected cubic graph with three vertices as centers of claws: it can have toughness 1, which is the lower bound of Proposition 1.

For r -regular graphs Chvátal gave examples of graphs which are r -regular and supertough, but these "inflations" are line graphs and hence $K(1,3)$ -free. We believe that almost every r -regular graph is supertough once r is sufficiently large. This would totally refute a conjecture in [5].

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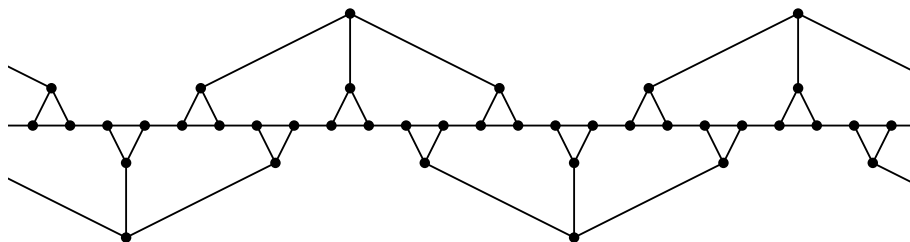


Figure 1: Part of the graph G_m .