# Broadcast Chromatic Numbers of Graphs

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**Abstract**

A function $\pi : V \to \{1, \ldots, k\}$ is a *broadcast coloring of order* $k$ if $\pi(u) = \pi(v)$ implies that the distance between $u$ and $v$ is more than $\pi(u)$. The minimum order of a broadcast coloring is called the *broadcast chromatic number* of $G$, and is denoted $\chi_b(G)$. In this paper we introduce this coloring and study its properties. In particular, we explore the relationship with the vertex cover and chromatic numbers. While there is a polynomial-time algorithm to determine whether $\chi_b(G) \leq 3$, we show that it is NP-hard to determine if $\chi_b(G) \leq 4$. We also determine the maximum broadcast chromatic number of a tree, and show that the broadcast chromatic number of the infinite grid is finite.

## 1 Introduction

The United States Federal Communications Commission has established numerous rules and regulations concerning the assignment of broadcast frequencies to radio stations. In particular, two radio stations which are assigned the same broadcast frequency must be located sufficiently far apart so that neither broadcast interferes with the reception of the other. The

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geographical distance between two stations which are assigned the same frequency is, therefore, directly related to the power of their broadcast signals.

These frequency, or channel, assignment regulations have inspired a variety of graphical coloring problems. One of these is the well-studied $L(2,1)$-coloring problem [2]. Let $d(u,v)$ denote the distance between vertices $u$ and $v$, and let $e(u)$ denote the eccentricity of $u$. Given a graph $G=(V,E)$, an $L(2,1)$-*coloring* is a function $c: V \to \{0,1,\ldots\}$ such that (i) $d(u,v)=1$ implies $|c(u)-c(v)| \geq 2$, and (ii) $d(u,v)=2$ implies $|c(u)-c(v)| \geq 1$. For a survey of frequency assignment problems, see [4].

In a similar way, Dunbar et al. [1] define a function $b: V \to \{0,1,\ldots\}$ to be a *dominating broadcast* if for every $u \in V$ (i) $b(u) \leq e(u)$, and (ii) $b(u)=0$ implies there exists a vertex $v \in V$ with $b(v) > 0$ and $d(u,v) \leq b(v)$. A broadcast is called *independent* if $b(u)=b(v)$ implies that $d(u,v) > b(u)$; that is, broadcast stations of the same power must be sufficiently far apart so that neither can hear each other's broadcast.

In this paper we introduce a new type of graph coloring. A function $\pi: V \to \{1,\ldots,k\}$ is called a *broadcast coloring of order* $k$ if $\pi(u)=\pi(v)$ implies that $d(u,v) > \pi(u)$. The minimum order of a broadcast coloring of a graph $G$ is called the *broadcast chromatic number*, and is denoted by $\chi_b(G)$. Equivalently, a broadcast coloring is a partition $\mathcal{P}_\pi = \{V_1, V_2, \ldots, V_k\}$ of $V$ such that each color class $V_i$ is an $i$-packing (pairwise distance more than $i$ apart). Note that in particular, every broadcast coloring is a proper coloring. Also, if $H$ is a subgraph of $G$, then $\chi_b(H) \leq \chi_b(G)$.

Throughout this article, we assume that graphs are simple: no loops or multiple edges. For terms and concepts not defined here, see [3]. In particular, we shall use the following notation: $\alpha_0(G)$ for the vertex cover number, $\beta_0(G)$ for the independence number, $\chi(G)$ for the chromatic number, $\omega(G)$ for the clique number, and $\rho_r(G)$ for the largest cardinality of an $r$-packing.

## 2 Basics

Every graph $G$ of order $n$ has a broadcast coloring of order $n$, since one can assign a distinct integer between 1 and $n$ to each vertex in $V$. There is a better natural upper bound.

**Proposition 2.1** *For every graph $G$,*

$$\chi_b(G) \leq \alpha_0(G) + 1,$$

*with equality if $G$ has diameter two.*

**Proof.** For the upper bound, give color 1 to every vertex in a maximum independent set in $G$. Then give every other vertex a distinct color. Since $n - \beta_0(G) = \alpha_0(G)$ by Gallai's theorem, the result follows.

If a graph has diameter two, then no two vertices can receive the same color $i$, for any $i \geq 2$. On the other hand, since the vertices which receive the color 1 form an independent set, there are at most $\beta_0(G)$ such vertices. □

From this it follows that the complete, the complete multipartite graphs, and the wheels have broadcast chromatic number one more than their vertex cover number. It also follows that computing the broadcast chromatic number is NP-hard (since vertex cover number is NP-hard for diameter 2).

If a graph is bipartite and has diameter three, then there is also near equality in Proposition 2.1.

**Proposition 2.2** *If $G$ is a bipartite graph of diameter 3, then $\alpha_0(G) \leq \chi_b(G) \leq \alpha_0(G) + 1$.*

**Proof.** By the diameter constraint, each color at least 3 appears at most once. Since the graph is bipartite of diameter 3, color 2 cannot be used twice on the same partite set, and thus can be used only twice overall. □

For an example of equality in the upper bound, consider the 6-cycle; for the lower bound, consider the 6-cycle where two antipodal vertices have been duplicated.

We now present the broadcast chromatic numbers for paths and cycles.

**Proposition 2.3** *For $2 \leq n \leq 3$, $\chi_b(P_n) = 2$; and for $n \geq 4$, $\chi_b(P_n) = 3$.*

**Proof.** The first $n$ entries in the pattern below represent a broadcast coloring for $P_n$.

$$1 \quad 2 \quad 1 \quad 3 \quad 1 \quad 2 \quad 1 \quad 3 \quad \ldots$$

This is clearly best possible. □

**Proposition 2.4** *For $n \geq 3$, if $n$ is 3 or a multiple of 4, then $\chi_b(C_n) = 3$; otherwise $\chi_b(C_n) = 4$.*

**Proof.** Since the cycle contains either $P_4$ or $K_3$, $\chi_b(C_n) \geq 3$. Let $v_0, v_1$, …, $v_{n-1}, v_0$ be the vertices of $C_n$ and suppose there is a broadcast coloring of order 3 with $n \geq 4$.

Then there cannot be two consecutive vertices neither of which has color 1. For suppose that $v_2$ has color 2 and $v_3$ has color 3, say. Then neither $v_0$ nor $v_1$ can receive color 2 or 3, and only one can receive color 1, a contradiction. Since vertices with color 1 cannot be consecutive, it follows that the vertices with color 1 alternate. In particular, $n$ is even.

Say the even-numbered vertices have color 1. But then no two consecutive odd-numbered vertices can receive the same color, and so they must alternate between colors 2 and 3. In particular, $n$ is a multiple of 4. It follows that if $n$ is not a multiple of 4, then $\chi_b(C_n) \geq 4$.

We consider now optimal broadcast colorings. For cycles with order $n$ a multiple of 4, the pattern

$$1, 2, 1, 3, 1, 2, 1, 3, \ldots, 1, 2, 1, 3$$

is a broadcast coloring. When $n$ is not a multiple of 4, the pattern consists of repeated blocks of "1,2,1,3" with an adjustment at the very end:

$$\begin{aligned} n = 4r+1: &\quad 1, 2, 1, 3, 1, 2, 1, 3, \ldots, 1, 2, 1, 3, 4 \\ n = 4r+2: &\quad 1, 2, 1, 3, 1, 2, 1, 3, \ldots, 1, 2, 1, 3, 1, 4 \\ n = 4r+3: &\quad 1, 2, 1, 3, 1, 2, 1, 3, \ldots, 1, 2, 1, 3, 1, 2, 4 \end{aligned}$$

Hence, if $n$ is a multiple of 4, then $\chi_b(C_n) \leq 3$; otherwise $\chi_b(C_n) \leq 4$. □

Just as the natural upper bound involves the vertex cover number, the natural lower bound involves the chromatic number.

**Proposition 2.5** *For every graph $G$,*

$$\omega(G) \leq \chi(G) \leq \chi_b(G).$$

It would be nice to characterize those graphs where the broadcast chromatic number is equal to the clique number. It is certainly necessary that the neighbors of any maximum clique form an independent set, and at least one vertex of such a clique has no neighbors outside the clique (so it can receive color 1). If the graph $G$ is a split graph, then this necessary condition is sufficient. (Recall that a *split graph* is a graph whose vertex set can be partitioned into two sets, $A$ and $B$, where $A$ induces a complete subgraph and $B$ is an independent set.)

On the other hand, a necessary condition for $\chi_b(G) = \chi(G)$ is that the clique number be large.

**Proposition 2.6** *For every graph $G$, if $\chi_b(G) = \chi(G)$ then $\omega(G) \geq \chi(G) - 2$.*

**Proof.** For, assume $\chi_b(G) = \chi(G) = m$. Consider the broadcast coloring as a proper coloring. If one can reduce the color of every vertex colored $m$ while still maintaining a proper coloring, then one has a contradiction. So there exists a vertex $v_m$ that has a neighbor of each smaller color: say $v_1, \ldots, v_{m-1}$ with $v_i$ having color $i$ ($v_i$ is unique for $i \geq 2$). Now, if one can reduce the color of the vertex $v_{m-1}$, this too will enable one to lower the color of $v_m$. It follows that $v_{m-1}$ has a neighbor of each smaller color. By the properties of the broadcast coloring, that neighbor must be $v_i$ for $i \geq 3$. By repeated argument, it follows that the vertices $v_3, v_4, \ldots, v_m$ form a clique. □

In another direction, we note that if one has a broadcast coloring, then one can choose any one color class $V_i$ to be a maximal $i$-packing. (Recolor vertices far away from $V_i$ with color $i$ if necessary.) However, one cannot ensure that all color classes are maximal $i$-packings.

## 3 Graphs with Small Broadcast Chromatic Number

We show here that there is an easy algorithm to decide if a graph has broadcast chromatic number at most 3. In contrast, it is NP-hard to determine if the broadcast chromatic number is at most 4. We start with a characterization of graphs with broadcast chromatic number 2.

**Proposition 3.1** *For any connected graph $G$, $\chi_b(G) = 2$ if and only if $G$ is a star.*

**Proof.** We know that the star $K_{1,m}$ has broadcast chromatic number 2. So assume $\chi_b(G) = 2$. Then $G$ does not contain $P_4$ and $\mathrm{diam}(G) \leq 2$. By Proposition 2.1, it follows that $\alpha_0(G) = 1$; that is, $G$ is a star. □

As regards those graphs with $\chi_b(G) = 3$, we start with a characterization of the blocks with this property. If $G$ is a graph, then we denote by $S(G)$ the *subdivision graph* of $G$, which is obtained from $G$ by subdividing every edge once. In $S(G)$ the vertices of $G$ are called the *original* vertices; the other vertices are called *subdivision* vertices.

**Proposition 3.2** *Let $G$ be a 2-connected graph. Then $\chi_b(G) = 3$ if and only if $G$ is either $S(H)$ for some bipartite multigraph $H$ or the join of $K_2$ and an independent set.*

**Proof.** Assume that $\chi_b(G) = 3$. Let $\pi : V(G) \to \{1, 2, 3\}$ be a broadcast coloring and let $V_i = \pi^{-1}(i)$, for $1 \leq i \leq 3$. It follows that $V_i$ is an $i$-packing for each $i$.

Let $v \in V_1$. The set $V_1$ is an independent set, so $N(v) \subseteq V_2 \cup V_3$. But $v$ has at most one neighbor in $V_2$, since $V_2$ is a 2-packing. Similarly, $v$ has at most one neighbor in $V_3$. Since $v$ has at least two neighbors, it follows that $v$ has degree 2 and is adjacent to exactly one vertex in each of $V_2$ and $V_3$.

It follows from Proposition 2.4 that the length of any cycle in $G$ is either 3 or a multiple of 4. Assume $G$ contains a triangle $\{x, y, z\}$. These vertices receive different colors; say $x \in V_2$ and $y \in V_3$. Then for every neighbor $t$ of $x$ apart from $y$, it follows that $t \in V_1$ and (since it is too close to $y$ to have another neighbor of color 3) that $N(t) = \{x, y\}$. Since $y$ cannot be a cut-vertex, it follows that this is the whole of $G$.

So assume that every cycle length is a multiple of 4. By the proof of Proposition 2.4, in any broadcast coloring of order 3 of a cycle, every alternate vertex receives color 1. It follows that $V_2 \cup V_3$ is an independent set. (Since $G$ is a block, every edge lies in a cycle.) In particular, $G$ is the subdivision of some multigraph $H$ where every subdvision vertex receives color 1. Furthermore, since $V_2$ and $V_3$ are 2-packings in $G$, they are independent sets in $H$; that is, $(V_2, V_3)$ is a bipartition of $H$.

Conversely, to broadcast color the subdivision of a bipartite multigraph, take $V_1$ as the subdivision vertices, and $(V_2, V_3)$ as the original bipartition. □

Now, in order to characterize general graphs with broadcast chromatic number 3, we define a *T-add to a vertex* $v$ as introducing a vertex $w_v$ and a set $X_v$ of independent vertices, and adding the edge $vw_v$ and some of the edges between $\{v, w_v\}$ and $X_v$. By extending the above result one can show:

**Proposition 3.3** *Let $G$ be a graph. Then $\chi_b(G) = 3$ if and only if $G$ can be formed by taking some bipartite multigraph $H$ with bipartition $(V_2, V_3)$, subdividing every edge exactly once, adding leaves to some vertices in $V_2 \cup V_3$, and then performing a single T-add to some vertices in $V_3$.*

Thus there is an algorithm for determining whether a graph has broadcast chromatic number at most 3. The key is that the colors 2 and 3 can seldom be adjacent. In particular, if vertices $u$ and $v$ are adjacent with $u$ with color 2 and $v$ with color 3, then any neighbor $a$ of $u$ apart from $v$ has $N(a) \subseteq \{u, v\}$. Apart from that, $V_2 \cup V_3$ must be an independent set, while every vertex of $V_1$ has degree at most 2. In particular, if two vertices with degree at least 3 are joined by a path of odd length, then at one of the ends of this path there must be two consecutive $V_2 \cup V_3$ vertices.

So a graph can be tested for having broadcast chromatic number 3 by identifying the places where $V_2$ and $V_3$ must be adjacent, coloring and trimming these appropriately, trimming leaves that are in $V_1$, and then seeing whether what remains with the partial coloring is a subdivision of a bipartite graph. We omit the details.

## 4 Intractable Colorings

In contrast to the above, the problem of determining whether a graph has a broadcast 4-coloring is intractable. We will need the following generalization. For a sequence of positive integers $s_1 \leq s_2 \leq \ldots \leq s_k$, an $(s_1, s_2, \ldots, s_k)$-*coloring* is a weak partition $\pi = (V_1, V_2, \ldots, V_k)$, where $V_j$ is an $s_j$-packing for $1 \leq j \leq k$. Then we define the decision problem:

> $(s_1, s_2, \ldots, s_k)$-Coloring
> Instance: Graph $G$
> Question: Does $G$ have an $(s_1, s_2, \ldots, s_k)$-coloring?

For example, a 3-coloring is a $(1, 1, 1)$-coloring. The Broadcast 4-Coloring problem is equivalent to the $(1, 2, 3, 4)$-Coloring problem. We will need the intractability of a special 3-coloring problem.

**Proposition 4.1** $(1, 1, 2)$-Coloring *is NP-hard.*

**Proof.** The proof is by reduction from normal 3-coloring. The reduction is to form $G'$ from $G$ as follows. Replace each edge $uv$ by the following: add a pentagon $P_{uv}$ and join $u, v$ to nonadjacent vertices of $P_{uv}$; add a pentagon $Q_{uv}$ and join $u, v$ to adjacent vertices of $Q_{uv}$ and add an edge joining two degree-two vertices of $Q_{uv}$. (See Figure 1.) The vertices $u, v$ are called *original* in $G'$.

We claim that $G'$ has a $(1, 1, 2)$-coloring iff $G$ is 3-colorable.

Assume that $G$ has a 3-coloring with colors red, blue and gold. We will 3-color $G'$ such that the gold vertices form a 2-packing. Start by giving the original vertices of $G'$ their color in $G$. For each two adjacent vertices $u, v$ of $G$, color gold one vertex from each of $P_{uv}$ and $Q_{uv}$ chosen as follows.

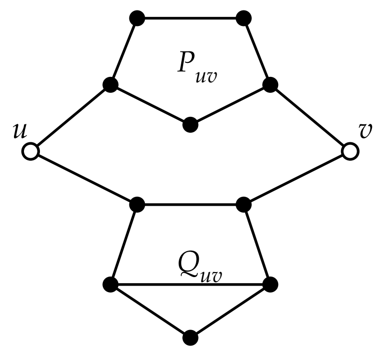


Figure 1: Replacing an edge $uv$

For $Q_{uv}$ it is the degree-2 vertex; for $P_{uv}$ it is the degree-2 vertex that is distance-3 from whichever of $u$ or $v$ is gold if one of them is gold, and it is the degree-2 vertex at distance 2 from both otherwise. Then it is easy to color the remaining vertices in $P_{uv}$ and $Q_{uv}$ with red and blue.

Conversely, suppose $G'$ has a $(1, 1, 2)$-coloring $\pi$ where the gold vertices form a 2-packing. Then adjacent vertices $u$ and $v$ cannot have the same color. For, if they are both red or both blue, then there is no possible coloring of $Q_{uv}$ (since one of the vertices in the triangle is gold); if they are both gold, then there is no possible coloring of $P_{uv}$. That is, restricted to $V(G)$, the coloring $\pi$ is a 3-coloring of $G$. $\Box$

**Theorem 4.2** Broadcast 4-Coloring *is NP-hard.*

**Proof.** We reduce from $(1, 1, 2)$-Coloring as follows. Given a connected graph $H$, form graph $H'$ by quadrupling each edge and then subdividing each edge. Thus, all original vertices have degree at least 4 and all subdivision vertices have degree 2.

The $(1, 1, 2)$-coloring of $H$ with red, blue and gold, becomes a broadcast 4-coloring $(V_1, V_2, V_3, V_4)$ of $H'$ by making $V_1$ all the subdivision vertices, $V_2$ all the red vertices, $V_3$ all the blue vertices and $V_4$ all the gold vertices. On the other hand, in a broadcast 4-coloring of $H'$, none of the original vertices can receive color 1. If we maximize the number of vertices receiving color 1, it follows that all subdivision vertices receive color 1, and all original vertices receive color 2, 3, or 4. So, the vertices colored 2 or 3 form an independent set in $H$ and the vertices colored 4 form a 2-packing in $H$. Thus we have a $(1, 1, 2)$-coloring of $H$. □

Comment: This proves that BROADCAST 4-COLORING is also NP-hard for planar graphs. It is an open question what the complexity is for cubic or 4-regular graphs. In another direction, it is easy to determine whether a graph is $(2, 2, 2)$-colorable (as only paths of any length and cycles of length a multiple of 3 are). But what is the complexity of $(1, 2, 2)$-COLORING?

## 5 Trees

We are interested in trees with large broadcast chromatic numbers. In order to prove the best possible general result, it is necessary to examine the small cases.

A tree of diameter 2 (that is, a star) has broadcast chromatic number 2. A tree of diameter 3 has broadcast chromatic number 3. The case of a tree of diameter 4 is more complicated, but one can still write down an explicit formula. We say that a vertex is *large* if it has degree 4 or more, and *small* otherwise. The key to the formula is the numbers of large and small neighbors of the central vertex.

**Proposition 5.1** *Let $T$ be a tree of diameter 4 with central vertex $v$. For $i = 1, 2, 3$, let $n_i$ denote the number of neighbors of $v$ of degree $i$, and let $L$ denote the number of large neighbors of $v$. If $L = 0$ then*

$$\chi_b(T) = \begin{cases} 4 & \text{if } n_3 \geq 2 \text{ and } n_1 + n_2 + n_3 \geq 3 \\ 3 & \text{otherwise,} \end{cases}$$

*and if $L > 0$ then*

$$\chi_b(T) = \begin{cases} L + 3 & \text{if } n_3 \geq 1 \text{ and } n_1 + n_2 + n_3 \geq 2 \\ L + 1 & \text{if } n_1 = n_2 = n_3 = 0 \\ L + 2 & \text{otherwise.} \end{cases}$$

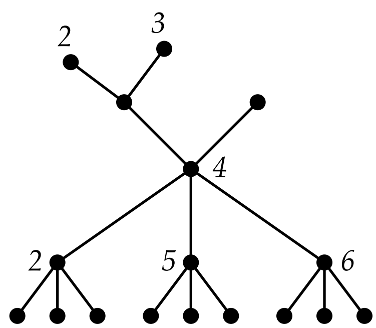


Figure 2: The tree $T_5$: the unlabeled vertices have color 1

**Proof.** To show the upper bound we need to exhibit a broadcast coloring.

A simple coloring is to color the center and the leaves not adjacent to it with color 1, and the remaining vertices with unique colors. This uses $L + n_3 + n_2 + n_1 + 1$ colors. It is optimal when $n_1 = n_2 = 0$ and either $2 \le n_3 \le 3$ and $L = 0$ or $0 \le n_3 \le 2$ and $L > 0$. (Note that $L + n_3 + n_2 \ge 2$ by the diameter condition.)

Another good coloring is as follows: put color 1 on the small neighbors of the center $v$ and on the children of large neighbors; put color 2 on one large neighbor (if one exists) and on one child of each small neighbor; put color 3 on the remaining children of small neighbors; and put unique colors on the remaining vertices. If $L = 0$, then this uses 4 colors if $n_3 > 0$ and 3 values otherwise. If $L > 0$, then this uses $L + 3$ colors if $n_3 > 0$ and $L + 2$ colors otherwise. This coloring is illustrated in Figure 2.

The only case not covered by the above two colorings is when $L = 0$ and $n_3 = 1$. In this case 3 colors suffice: use color 3 on the central vertex, color 2 on its degree-3 neighbor and the children of its degree-2 neighbors, and color 1 on the remaining vertices.

For a lower bound, proceed as follows. If the center is colored 1, then the coloring uses $L + n_3 + n_2 + n_1 + 1$ colors, which is at least the above bound. So we may assume that the center is not colored 1.


Figure 3: The smallest trees with $\chi_b(T) = 4$

If a large neighbor receives any color other than 2, then either it or one of its children receives a unique color. Thus we may remove it and induct. So we may assume that every large neighbor of the center receives color 2. This means there is at most one large neighbor.

At least three colors are always needed. For the case that $L = 0$, it is enough to argue that 4 colors are needed if $n_3 = 2$, $n_2 = 0$ and $n_1 = 1$ (as any other case contains this as a subgraph). (This is tree $A_8$ in Figure 3.) If any degree-3 vertex receives color 1, then three more colors are needed for its neighbors. On the other hand, the degree-3 vertices induce a $P_3$, and so require three new colors if 1 is not used.

In fact, this observation also takes care of the case where there is only one large neighbor. □

**Proposition 5.2** *The minimum order of a tree with broadcast chromatic number* 2 *is* 2*. For* 3 *it is* 4 *and for* 4 *it is* 8*. Furthermore,* $P_4$ *is the unique tree on* 4 *vertices that needs* 3 *colors. The two trees on 8 vertices that need* 4 *colors are (i) the diameter-*4 *tree with* $n_3 = 2$*,* $n_1 = 1$ *and* $L = n_2 = 0$*, called* $A_8$*; and (ii) the diameter-*6 *tree where the two central vertices have degree-*3 *and for each central vertex its three neighbors have degrees* 1*,* 2 *and* 3 *respectively, called* $B_8$*. (These are depicted in Figure 3.)*

**Proof.** By the above result, $A_8$ is the unique smallest tree with diameter 4 that needs four colors. So we need only examine the small trees with diameter 5 or more, which is easily done. □

In another direction there is an extension result.

**Proposition 5.3** *Let $T$ be a graph but not $P_4$. Suppose $T$ contains a path $t, u, v, w$ where $t$ has degree 1 and $u$ and $v$ have degree 2. Then $\chi_b(T) = \chi_b(T - t)$.*

**Proof.** Take an optimal broadcast coloring of $T - t$. Since $T \neq P_4$, $T - t$ contains $P_4$ and hence uses at least three colors.

If $u$ receives any color except 1, then one can color $t$ with 1. So assume $u$ receives color 1. If $v$ receives any color except 2, then one can color $t$ with color 2. So assume $v$ receives color 2. If $w$ receives any color except 3, then one can color $t$ with color 3. So assume $w$ receives color 3. But then one can recolor as follows: $v$ gets color 1, $u$ gets color 2, and $t$ gets color 1. □

For example, this shows that $\chi_b(P_n) = 3$ for all $n \geq 4$.

We are now ready to determine the maximum broadcast chromatic number of a tree. An extremal tree $T_d$ for $d \geq 2$ is constructed as follows: it has diameter 4; $n_1 = n_3 = 1$, $n_2 = 0$, $L = d - 2$ and all large vertices have degree exactly 4. The tree $T_d$ has $4d - 3$ vertices and $\chi_b(T_d) = d + 1$. The tree $T_5$ is shown in Figure 2.

**Theorem 5.4** *For all trees $T$ of order $n$ it holds that $\chi_b(T) \leq (n + 7)/4$, except when $n = 4$ or 8, when the bound is 1/4 more, and these bounds are sharp.*

**Proof.** We have shown sharpness above. The proof of the bound is by induction on $n$.

If $n \leq 8$ the result follows from Proposition 5.2. If $\operatorname{diam}(T) \leq 3$, then $\chi_b(T) \leq 3$. If $\operatorname{diam}(T) = 4$, then the bound follows from Proposition 5.1. So assume $n \geq 9$ and $\operatorname{diam}(T) \geq 5$.

Define a *penultimate* vertex as one with exactly one non-leaf neighbor. Suppose some penultimate $u$ has degree 4 or more. Define $T'$ to be the tree after the removal of $u$ and all its leaf-neighbors. Then take an optimal broadcast coloring of $T'$, and extend to a broadcast coloring of $T$ by giving $u$ a new unique color and its leaf neighbors color 1. By the inductive hypothesis it follows that $\chi_b(T) \leq \chi_b(T') + 1 \leq (n+3)/4 + 1 = (n+7)/4$, unless $T'$ is an exceptional tree.

But the three exceptional trees can be broadcast colored with 4 colors so that, for any specific vertex $v$, neither it nor any of its neighbors receives color 2. So let $v$ be $u$'s other neighbor and color $T'$ thus; then color $u$ with color 2 and its leaf-neighbors with color 1. So, in this case $\chi_b(T) \leq 4$, which establishes the bound.

So we may assume that every penultimate vertex has degree 2 or 3.

Now, define a *late* vertex as one that is not a penultimate, but at most one of its neighbors is not a penultimate or a leaf. (For example, the third-to-last vertex on a diametrical path.) For a late vertex $v$, define $T_v$ as the subtree consisting of $v$, all its penultimate neighbors, and any leaf adjacent to one of these. Since $\operatorname{diam}(T) \geq 5$, $T_v$ is not the whole of $T$; let $w$ be $v$'s other neighbor. Then define $T' = T - T_v$.

If $|T_v| = 3$, then $v$ has degree 2 and its penultimate neighbor has degree 2. So by the above result, $\chi_b(T) = \chi_b(T')$ and we are done. Therefore we may assume that $|T_v| \geq 4$.

Give $T'$ an optimal broadcast coloring. Then, if $w$ is not colored 3, one can extend this to a broadcast coloring of $T$ by giving $v$ a new unique color, all its neighbors in $T_v$ color 1 and the remaining vertices of $T_v$ colors 2 or 3. We are done by induction—one new color for at least four vertices—unless $|T_v| = 4$ and $T'$ is an exceptional tree. But in this case one can readily argue that $\chi_b(T) \leq 4$.

So assume the vertex $w$ receives color 3 in any coloring of $T'$. (In particular, this means that $T'$ is not one of the exceptional trees.) Now, we can afford to recolor $w$ with a new color and proceed as above if $|T_v| \geq 8$. So assume that $|T_v| \leq 7$.

If $v$ has only one penultimate neighbor of degree 3, then one can color $T_v$ with colors 1 and 2 except for $v$, and so are done. So we may assume that $v$ has two degree-3 neighbors. But then $v$ has exactly two neighbors in $T_v$ and these have degree 3. But then one can color $T_v$ with colors 1 and 2, except for one neighbor of $v$, with $v$ receiving color 1. And hence we are done by the inductive hypothesis. □

## 6 Grids

We will now investigate broadcast colorings of grids $G_{r,c}$ with $r$ rows and $c$ columns. The exact values for $r \leq 5$ are given in the following result

**Proposition 6.1** *$\chi_b(G_{2,c}) = 5$ for $c \geq 6$; $\chi_b(G_{3,c}) = 7$ for $c \geq 12$; $\chi_b(G_{4,c}) = 8$ for $c \geq 10$; and $\chi_b(G_{5,c}) = 9$ for $c \geq 10$. The values for smaller grids are as follows:*

| $m\backslash n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 4 | 4 | 5 | ... | | | | | | |
| 3 | | 4 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 7 | ... |
| 4 | | | 5 | 7 | 7 | 7 | 7 | 7 | 8 | ... | | |
| 5 | | | | 7 | 7 | 7 | 8 | 8 | 9 | ... | | |

**Proof.** Consider the following coloring pattern. For $c \geq 2$, the first $c$ columns indicate that the values for $\chi_b(G_{2,c})$ stated are in fact upper bounds.

$$\begin{array}{|ccc:ccc|c} 2 & 1 & 4 & 1 & 3 & 1 & \dots \\ 1 & 3 & 1 & 2 & 1 & 5 & \dots \end{array}$$

Consider the following coloring pattern. For $c \geq 3$, the first $c$ columns indicate that the values for $\chi_b(G_{3,c})$ stated are in fact upper bounds.

$$\begin{array}{|cccccc:cccccc|c} 2 & 1 & 3 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 3 & 1 & \dots \\ 1 & 4 & 1 & 5 & 1 & 6 & 1 & 4 & 1 & 5 & 1 & 7 & \dots \\ 3 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & \dots \end{array}$$

Consider the following coloring pattern. For $c \geq 4$, the first $c$ columns indicate that the values for $\chi_b(G_{4,c})$ stated are in fact upper bounds.

$$\begin{array}{|ccccc:ccccc|c} 1 & 2 & 1 & 3 & 1 & 2 & 1 & 3 & 1 & 6 & \dots \\ 3 & 1 & 5 & 1 & 7 & 1 & 4 & 1 & 2 & 1 & \dots \\ 1 & 4 & 1 & 2 & 1 & 3 & 1 & 5 & 1 & 8 & \dots \\ 2 & 1 & 3 & 1 & 6 & 1 & 2 & 1 & 3 & 1 & \dots \end{array}$$

Consider the following coloring pattern. Let $i$ and $j$ denote the row and column of a vertex, with $1 \le i \le r$ and $1 \le j \le c$. Then assign color 1 to every vertex with $i + j$ odd; assign color 2 to every vertex with $i$ and $j$ odd and $i + j$ not a multiple of 4; and assign color 3 to every other vertex with $i$ and $j$ odd. The picture looks as follows.

$$
\begin{array}{|cccc|cccc|c}
2 & 1 & 3 & 1 & 2 & 1 & 3 & 1 & \dots \\
1 & - & 1 & - & 1 & - & 1 & - & \dots \\
3 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & \dots \\
1 & - & 1 & - & 1 & - & 1 & - & \dots \\
2 & 1 & 3 & 1 & 2 & 1 & 3 & 1 & \dots
\end{array}
$$

(The uncolored vertices induce a copy of the grid in the square of the graph.) For $G_{5,c}$ the uncolored vertices should be colored as follows:

$$
\begin{array}{|cccc|cccc|c}
4 & 6 & 5 & 8 & 4 & 7 & 5 & 9 & \dots \\
5 & 7 & 4 & 9 & 5 & 6 & 4 & 8 & \dots
\end{array}
$$

It is to be noted that the patterns for $G_{5,8}$ and $G_{5,9}$ are exceptions.

Lower bounds in general can be verified by computer. Some can be verified by hand. One useful idea is the following. For the lower bound for $G_{2,c}$ where $c \ge 6$, note that any copy of $G_{2,3}$ contains a color greater than 3. If one considers the three columns after a column containing a 4, then that $G_{2,3}$ has at least one of its vertices colored 5 or greater. □

The following table provides some more upper bounds. An asterisk indicates an exact value.

| $m\backslash n$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 8* | 9* | 9* | 9* | 9* | 9* | 10 | 10 | 10 | 11 | 11 |
| 7 | | 9* | 9* | 10 | 10 | 11 | 11 | 11 | 11 | 12 | 12 |

Often, the greedy approach produces a bound close to optimal. By the time the grids have around 20 rows (together with several hundred columns), the greedy approach uses more than 25 colors. As the following theorem implies, these bounds are not best possible.

**Theorem 6.2** *For any grid $G_{m,n}$, $\chi_b(G_{m,n}) \leq 23$.*

**Proof.** There is a broadcast coloring of the infinite grid that uses 23 colors. The coloring is illustrated below. This provides a broadcast coloring of any finite subgrid.

As in the coloring of $G_{5,n}$ above, we start by coloring with 1s, 2s and 3s such that the uncolored vertices occur in every alternate row and column. Then the following coloring is used to tile the plane.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 5 | 8 | 4 | 5 | 9 | 4 | 5 | 8 | 4 | 5 | 9 |
| 10 | 6 | 11 | 7 | 12 | 6 | 10 | 7 | 11 | 6 | 13 | 7 |
| 5 | 4 | 9 | 5 | 4 | 8 | 5 | 4 | 9 | 5 | 4 | 8 |
| 14 | 7 | 15 | 6 | 13 | 7 | 16 | 6 | 17 | 7 | 12 | 6 |
| 4 | 5 | 18 | 4 | 5 | 11 | 4 | 5 | 19 | 4 | 5 | 11 |
| 20 | 6 | 21 | 7 | 10 | 6 | 14 | 7 | 15 | 6 | 10 | 7 |
| 5 | 4 | 8 | 5 | 4 | 9 | 5 | 4 | 8 | 5 | 4 | 9 |
| 13 | 7 | 11 | 6 | 22 | 7 | 12 | 6 | 11 | 7 | 23 | 6 |
| 4 | 5 | 9 | 4 | 5 | 8 | 4 | 5 | 9 | 4 | 5 | 8 |
| 12 | 6 | 10 | 7 | 15 | 6 | 13 | 7 | 10 | 6 | 14 | 7 |
| 5 | 4 | 17 | 5 | 4 | 11 | 5 | 4 | 18 | 5 | 4 | 11 |
| 16 | 7 | 19 | 6 | 14 | 7 | 20 | 6 | 21 | 7 | 15 | 6 |

The coloring was found by placing the colors 4 through 9 in a specific pattern, and then using a computer to place the remaining colors. $\square$

Schwenk [5] has shown that the broadcast chromatic number of the infinite grid is at most 22.

## 7 Other Grid-like Graphs

While the broadcast chromatic number of the family of grids is bounded, this is not the case for the cubes. We start with a simple result on the cartesian product $\square$ with $K_2$.

**Proposition 7.1** *If* $\chi_b(G) \geq \text{diam}(G)+x$, *then* $\chi_b(G\Box K_2) \geq \text{diam}(G\Box K_2)+2x-1$.

**Proof.**

The result is true for $x \leq 0$, since $\chi_b(G\Box K_2) \geq \chi_b(G)$. So assume $x \geq 1$. Suppose there exists a broadcast coloring $\pi$ of $G\Box K_2$ that uses at most $\text{diam}(G\Box K_2)+2x-2$ colors. It follows that the $2x-1$ biggest colors—call them $Z$—are used at most once in $G\Box K_2$. It follows that one of the copies of $G$ is broadcast-colored by the colors up to $\text{diam}(G\Box K_2)-1 = \text{diam}(G)$ together with at most $x-1$ colors of $Z$. Thus $\chi_b(G) \leq \text{diam}(G)+x-1$, a contradiction. □

For example, this shows that the broadcast chromatic number of the cube is at least a positive fraction of its order. Next is a result regarding the first few hypercubes $Q_d$.

**Proposition 7.2** $\chi_b(Q_1) = 2$, $\chi_b(Q_2) = 3$, $\chi_b(Q_3) = 5$, $\chi_b(Q_4) = 7$, *and* $\chi_b(Q_5) = 15$.

**Proof.** The values for $Q_1$ and $Q_2$ follow from earlier results. Consider $Q_3$. The upper bound is from Proposition 2.1. To see that five colors are required, note that since $\beta_0(Q_3) = 4$, at most four vertices can be colored 1. Further, at most two vertices can be colored 2; but, if four vertices are colored 1 then no more than one vertex can be colored 2. Therefore, the number of vertices colored 1 or 2 must be at most five. Since $\text{diam}(Q_3) = 3$, no color greater than 2 can be used more than once. As there are eight vertices in $Q_3$, this means that at least five colors are required. The lower bound for $Q_4$ follows from the value for $Q_3$ by the above proposition.

For a suitable broadcast coloring in each case, use the greedy algorithm as follows. Place color 1 on a maximum independent set; then color with color 2 as many as possible, then color 3 and so on. □

We look next at the asymptotics. Bounds for the packing numbers of the hypercubes are well explored in coding theory. For our purposes it suffices to note the bounds:

$$\rho_j(Q_k) \leq \frac{2^k}{\sum_{i=0}^{\lfloor j/2 \rfloor} \binom{k}{i}}.$$

and $\rho_2(Q_k) \geq 2^{k-1}/k$ by, for example, the (computer) Hamming code.

**Proposition 7.3** $\chi_b(Q_k) \sim (\frac{1}{2} - O(\frac{1}{k}))2^k$.

**Proof.** For the upper bound, color a maximum independent set with color 1. Then color as many vertices with color 2 as possible. Clearly one can choose at least half a maximum 2-packing. And then use unique colors from there on.

The lower bound is from the packing bounds above, together with the fact that $\beta_0(Q_k) = 2^{k-1}$. □

For the sake of interest, the following table gives some bounds computed by using these approaches.

| $n$ | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|
| $\chi_b(Q_n) \geq$ | 15 | 28 | 63 | 132 | 285 | 610 |
| $\chi_b(Q_n) \leq$ | 25 | 49 | 95 | 219 | 441 | 881 |

We consider next another relative of the grid. Define a path of thickness $w$ as the lexicographic product $P_n[K_w]$, that is, every vertex of the path is replaced by a clique of $w$ vertices. A broadcast coloring of a thick path is equivalent to assigning $w$ distinct colors to each vertex of the path and requiring a broadcast coloring.

Let $g(w)$ denote the broadcast chromatic number of the infinite path of thickness $w$. A natural approach is to assign one color to every vertex, then a second and so on. So the following parameter arises naturally. Let $f(m)$ denote the broadcast chromatic number of the infinite path given that no color smaller than $m$ is used.

**Proposition 7.4** *For all $m$ sufficiently large, $f(m) \leq 3m - 1$. Indeed, for all $m$ $f(m) \leq 3m + 2$.*

**Proof.** This result is easy to verify for small $m$. For the cases up to $m = 33$ a computer search produced a suitable coloring.

The bound $f(m) \leq 3m - 1$ is by induction on $m$. The base case is $m = 34$, where it can be checked by computer that the greedy algorithm eventually settles into a cycle of length 176400 moves and uses no color more than 101.

In general, take the broadcast coloring for $f(m)$. Then replace the vertices of color $m$ with the three colors in succession $3m, 3m+1, 3m+2$. The result is still a broadcast coloring. □

So if we have the thick path, it follows that $g(w) \leq (1 + o(1))3^w$. The idea is to fill the first level, then the second and so on.

As for lower bounds, it holds for the path that at most $1/(i+1)$ of the vertices can receive color $i$. Hence for $H(s,t) = \sum_s^t 1/i$ it follows that we need $t$ such that $H(m,t) \leq 1$. By standard estimates for the harmonic series, $H(1,t) \sim \ln(t) - \gamma$, where $\gamma$ is the Euler constant. Thus $f(m) \geq em - o(m)$.

As for $g(w)$, similar considerations imply that we need $H(1,t) \geq w$ so that $g(w) \geq \Omega(e^w)$. It is unclear if this is the correct order of magnitude. We note that we quickly run into problems with achieving a proportion of $1/(i+1)$ for any beyond the first $w$ colors.

# 8 Open Problems

1. Can the broadcast chromatic number of a tree be computed in polynomial time?
2. What is the maximum broadcast chromatic number of a grid (and when is it first obtained)?
3. What about other “grids” such as the three-dimensional grid or the hexagonal lattice?
4. What is the maximum broadcast chromatic number of a cubic graph on $n$ vertices?

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