Acyclic Colorings of Planar Graphs

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Abstract

It is shown that a planar graph can be partitioned into three linear forests. The sharpness of the result is also considered.

In 1969, Chartrand and Kronk [2] showed that the vertex arboricity of a planar graph is at most 3. In other words, the vertex set of a planar graph can be partitioned into three sets each inducing a forest. In this paper we present an improvement on this result: that the vertex set of a planar graph can be partitioned into three sets such that each set induces a *linear forest.* A linear forest is one in which every component is a path. We will, for brevity, call such a partition a 3LF-coloring.

This result establishes a conjecture of Broere and Mynhardt [1]. It also improves upon a result of Cowen, Cowen and Woodall [3] that one is guaranteed a partition into three sets each inducing a graph of maximum degree at most two. Our result is proved by a simple extension of the techniques in [3].

Theorem 1 *The vertex set of any planar graph can be partitioned into three sets such that each set induces a linear forest.*

Proof. We prove by induction on the number of vertices that:

> every planar graph can be partitioned as above, with the added requirement that any two specified adjacent vertices are properly colored.

This is true for graphs without edges. So let G be a planar graph with specified adjacent vertices u and v. If one of u and v has degree two or less, then a simple inductive argument will establish the result. So we may assume that u and v have degree at least three.

Consider an embedding of G in the plane. Insert a new vertex x subdividing the edge uv. Further, while preserving planarity, add edges to ensure that there exists a cycle that separates u from v. Let W be such a cycle of minimum length in the resultant planar graph G'. Then $x \in W$ and W is chord-free, so that $W - x$ induces a path in G'.

Let G_1 (G_2) consist of W and all vertices and edges of G' inside (outside) W; say $u \in G_1$ and $v \in G_2$. Let G'_1 (G'_2) be obtained from G_1 (G_2) by contracting W into a single new vertex w. These are both planar graphs with fewer vertices than G, and w is adjacent to u (v) in G'_1 (G'_2).

By the inductive hypothesis, there is a 3LF-coloring of G'_1 (G'_2) such that u and w (v and w) are properly colored. We may assume that w receives the same color in both these colorings, and that u and v receive different colors. We now transfer these colors back to G, giving all the vertices of $W-x$ the color received by w. This yields a 3LF-coloring in which u and v are properly colored, as desired. QED

We now consider the sharpness of this result.

Theorem 2 *Given a planar graph, one is guaranteed neither*
a) a partition into two linear forests and a matching; nor
b) a partition into three linear forests such that every pair of colors induce an outerplanar graph.

PROOF. **a)** We will construct for k a positive integer, a planar graph G_k such that, for any 3LF-coloring, the subgraph induced by each color class must contain a path on at least k vertices.

Start with a K_3 with vertex set $\{v_1, v_2, v_3\}$. For each pair of vertices v_i and v_j, introduce a long path P_{ij} such that every vertex of P_{ij} is adjacent to v_i and v_j. Let the number of vertices on P_{ij} be the maximum of $5k$ and 22. Then inside each triangle consisting of a vertex of the K_3, say v_i, and two consecutive vertices from P_{ij}, say w_1 and w_2, introduce a path $Q(v_i, w_1, w_2)$ on k vertices such that every vertex of $Q(v_i, w_1, w_2)$ is adjacent to v_i and to w_1. This yields the planar graph G_k.

Now, consider any 3LF-coloring of G_k. As v_1, v_2 and v_3 induce a cycle, some two vertices v_i and v_j receive different colors, say A and B respectively. As v_i and v_j both have at most two neighbors of their own color, P_{ij} must be predominantly the third color C. Indeed, it must have at least k consecutive vertices from C.

As P_{ij} also has at least 22 vertices, it has at least three pairs of consecutive (non-terminal) vertices such that they, and both their neighbors on P_{ij}, have color C. As v_i has at most two neighbors of color A, there exists such a pair of consecutive vertices w_1 and w_2 where v_i has no neighbor of color A inside the triangle formed by v_i, w_1 and w_2. But w_1 and w_2 both have two neighbors of color C on P_{ij}. Thus all the vertices of the path $Q(v_i, w_1, w_2)$ inside this triangle receive color B. Similarly, by considering v_j we may find vertices w'_1 and w'_2 on P_{ij} such that all the vertices of the path $Q(v_j, w'_1, w'_2)$ receive color A.

b) For any graph G we define G^* as that graph formed by adding, between each pair of adjacent vertices, five internally disjoint paths each of length two. Note that if G is planar then so is G^*.

Now, let $G = K_4$ and consider a 3LF-coloring of G^*. Then there exist two adjacent vertices of G, x and y say, which receive the same color A say. In G^*, x and y have (at least) five mutual neighbors; none of these neighbors can have color A, so at least three receive color B say. But then we have a $K(2,3)$ in the graph induced by $A \cup B$, and thus A and B do not induce an outerplanar graph. QED

Part (b) shows that Conjecture A of [3] is false. Indeed, as a consequence of part (a) and the construction of part (b) (i.e. by considering G_k^*), it follows that there are graphs which, for any 3LF-coloring, can have at most one pair of colors inducing an outerplanar graph. The following is still open, though.

Conjecture. [3] *The vertex set of any planar graph can be partitioned into two sets such that one set induces an outerplanar graph and the other a linear forest.*

The above conjecture, if true, would strengthen Theorem 1, as an outerplanar graph can be partitioned into two linear forests (cf. [1]). Further, an outerplanar graph can be partitioned into a forest and an independent set (via a simple inductive argument, say). Thus the conjecture would imply that any planar graph can be partitioned into a forest, a linear forest, and an independent set. This would improve on the result of Stein [4] that a planar graph can be partitioned into two forests and an independent set. Wegner [5] showed that one cannot guarantee a forest and two independent sets.

As a final comment, we observe that the constructions used in the proof of Theorem 2 yield graphs with many cut-triangles. It is therefore open whether one can obtain stronger results for 4-connected planar graphs.

References

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