NOTE

Mistilings with Dominoes

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Abstract

We consider placing dominoes on a checker board such that each domino covers exactly some number of squares. Given a board and a type of domino, we define the mistiling ratio as the minimum proportion of squares that are covered in a maximal packing of the board with dominoes of that type. In this note we determine and bound the mistiling ratio for some types of dominoes on the infinite checker board.

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1 Introduction

People often study the best way to pack an object. But Sands [4] considered the following problem. What is the minimum number of dominoes that can be placed on an $n \times m$ checker board, each covering two adjacent squares, such that there is no room for another domino? Sands showed that if nm is divisible by 3 then the answer is $nm/3$.

This question was subsequently discussed by Pearce [3] and Gardner [1]. Recently Gyárfás, Lehel and Tuza [2] proved results for checker boards as well as for dominoes which cover two adjacent squares on other boards.

In this note we consider Sands' question extended to different dominoes. We consider dominoes which cover exactly some number of squares on the board. Given a type of domino, we define a *tiling* of a board as an arrangement of non-overlapping dominoes such that no further domino can be fitted onto the board. The *mistiling ratio* of the board is the proportion of squares covered by the worst tiling. Here we determine the mistiling ratio on infinite boards for the three-square hook domino and arbitrary square dominoes, and give bounds for the ratio for longer dominoes.

With this terminology, Sands and Gyárfás et al. showed that:

Theorem 1 [4, 2] *The mistiling ratio for the standard* 1×2 *domino on the infinite board is* $2/3$*. It is also* $2/3$ *for an* $n \times m$ *board when* nm *is divisible by* 3.

It is not hard to show that this theorem extends to boards in arbitrary dimensions (where each domino occupies two abutting "squares"). We omit the proof.

2 Hooks

A *hook* is a domino on three squares formed by gluing a square onto the side of the standard domino. An inefficient tiling is shown in Figure 1. This is worst possible:

Theorem 2 *The mistiling ratio of the hook on the infinite board is* $6/11$.

PROOF. The above construction shows that the mistiling ratio is at most $6/11$. So we must prove the lower bound. Consider a tiling and an $n \times n$ region $\mathcal{R}$ which has D full dominoes. Each of the $(n-1)^2$ possible 2×2 subregions of $\mathcal{R}$ has at least two squares covered. Moreover, if it contains a whole domino then a 2×2 subregion has at least three squares covered. So at least $3D + 2((n-1)^2 - D)$ squares of $\mathcal{R}$ are covered, where most coverings are counted four times. The actual number of squares covered is $3D + O(n)$. Hence $4(3D + O(n)) \geq 3D + 2((n-1)^2 - D)$, whence $D \geq 2n^2/11 - O(n)$. So the mistiling ratio, which is at least the limit as $n \to \infty$ of $(3D)/n^2$, is lower bounded by $6/11$. □

3 Squares

An inefficient tiling by square $m \times m$ dominoes is obtained by placing the dominoes with lower left corners on the $(a(2m-1), b(2m-1))$ squares for $a, b \in \mathbb{Z}$. See Figure 2. The next theorem shows that this is worst possible:

Theorem 3 *The mistiling ratio of the* $m \times m$ *square domino on the infinite board is* $m^2/(2m-1)^2$.

PROOF. The above construction shows that the mistiling ratio is at most this value. So we must prove the lower bound. Consider a tiling with $m \times m$ dominoes. We claim that for any $r \times s$ region $\mathcal{R}$, with $m - 1 \leq r, s \leq 2m - 1$, at least $(r - m + 1)(s - m + 1)$ of its squares are covered. This implies that the mistiling ratio is at least $m^2/(2m-1)^2$.

If $r = m - 1$ or $s = m - 1$ the claim is trivial. So assume $r, s \geq m$. If one domino is completely inside $\mathcal{R}$ then we are done; so assume otherwise. If the top row of $\mathcal{R}$ has at most $m - 1$ uncovered squares, then we may apply induction on the remaining $(r-1) \times s$ region to deduce that the number of squares of $\mathcal{R}$ covered is at least $[s - (m-1)] + [((r-1) - m + 1)(s - m + 1)]$. So we may assume that both rows and both columns on the edge of $\mathcal{R}$ have at least m uncovered squares. There are thus at most four dominoes that extend onto $\mathcal{R}$, each covering one of the corners. We identify the four possible dominoes by their geographic positions.

Now, consider trying to slide a new domino d_1 onto $\mathcal{R}$, between the SW and NW dominoes (if any) from the west. (There are at least m rows between the SW and NW dominoes.) The domino d_1 must be blocked before it is fully on $\mathcal{R}$; say it is blocked by the SE domino. Similarly a domino d_2 slid on from the east, along the SE one, must eventually be blocked by the NW domino. This also means that if we slid domino d_1 along the NW domino, it would be blocked by the SE domino.

Let a (resp. b) denote the number of rows (columns) covered by the NW domino, and let c (resp. d) denote the number of rows (columns) covered by the SE domino. Then $a + c \geq r - m + 1$, since domino d_1 cannot slide between the SE and NW dominoes. Also $d \geq s - m + 1$ since the SE domino blocks d_1 from getting fully onto $\mathcal{R}$, and $b \geq s - m + 1$ since the NW domino blocks d_2. The SE and NW dominoes together cover $ab + cd$ squares of the region $\mathcal{R}$, which proves the claim and completes the proof of the theorem. □□

4 Longs

Here we consider tilings of the infinite board with $1 \times m$ dominoes for $m \geq 3$. An inefficient tiling is shown in Figure 3. We conjecture that this is a mistiling. For $m = 3$ there is another tiling with the same ratio: with all the dominoes horizontal. A simple lower bound is the following:

Theorem 4 *The mistiling ratio of the* $1 \times m$ *domino on the infinite board is at least* $2/(m + 1)$.

Proof. Consider a tiling and an $n \times n$ region $\mathcal{R}$ which has D full dominoes and H uncovered squares. Let M denote the number of pairs (H', D') where H' is an uncovered square and D' is the first domino one encounters when one moves directly down from H', with both H' and D' completely inside $\mathcal{R}$. Each uncovered square, except one near the bottom of $\mathcal{R}$, is in exactly one such pair. So $M = H - O(n)$. A vertical domino can be in at most $m-1$ pairs. And it can be shown that a horizontal domino is in at most $(m-1)^2$ pairs. Without loss of generality, we may assume that at least half of the dominoes are vertical. Thus $M \leq (m - 1)D/2 + (m - 1)^2 D/2$. Hence

$$Dm(m - 1)/2 \geq M \geq H - O(n) = n^2 - mD - O(n),$$

whence $D \geq 2n^2/(m(m + 1)) - O(n)$. So the mistiling ratio, which is at least the limit as $n \to \infty$ of $(mD)/n^2$, is lower bounded by $2/(m + 1)$. □□

For $m \geq 3$ one can improve on this by showing that many dominoes are not in as many pairs as the bounds suggest. Using this approach we can show that the average domino is in at most $(m-1)^2/2 + (m-1)/4$ pairs, and thus the mistiling ratio is at least $4m/(2m^2+m+1)$. But we believe that:

Conjecture 1 *For $m \geq 3$ the mistiling ratio of the $1 \times m$ domino on the infinite board is $2m/(m^2+1)$.*

For example, we conjecture that the mistiling ratio for the 1×3 domino is $3/5$, but can only show that it is at least $6/11$.

References

[1] M. Gardner, Chapter 15 of *Knotted Doughnuts and Other Mathematical Entertainments*, Freeman, 1986.

[2] A. Gyárfás, J. Lehel & Zs. Tuza, Clumsy packing of dominos, *Discrete Math.* **71** (1988) 33–46.

[3] M. Pearce, *Games and Puzzles*, November 1973.

[4] B. Sands, The Gunport Problem, *Math. Magazine* **44** (1971) 193–196.

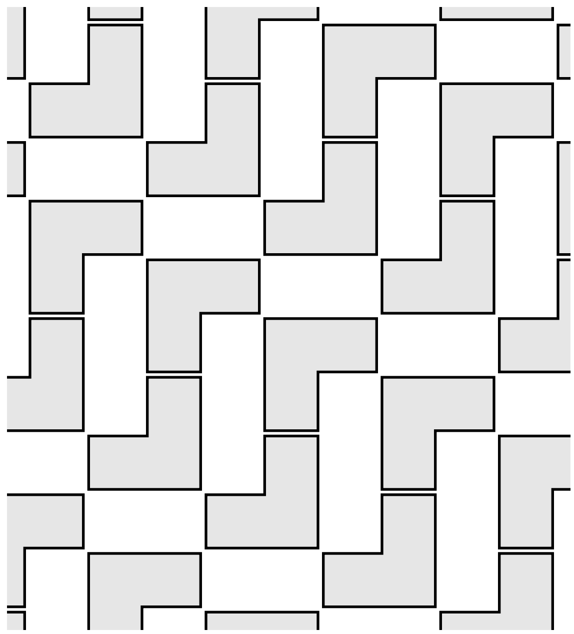

Figure 1: A Mistiling with Hooks

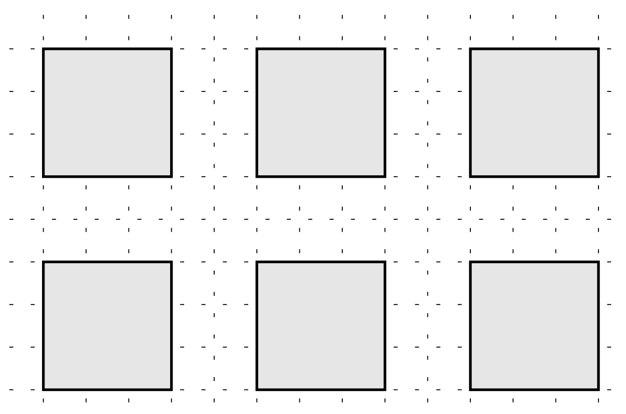

Figure 2: A Mistiling with 3×3 Squares

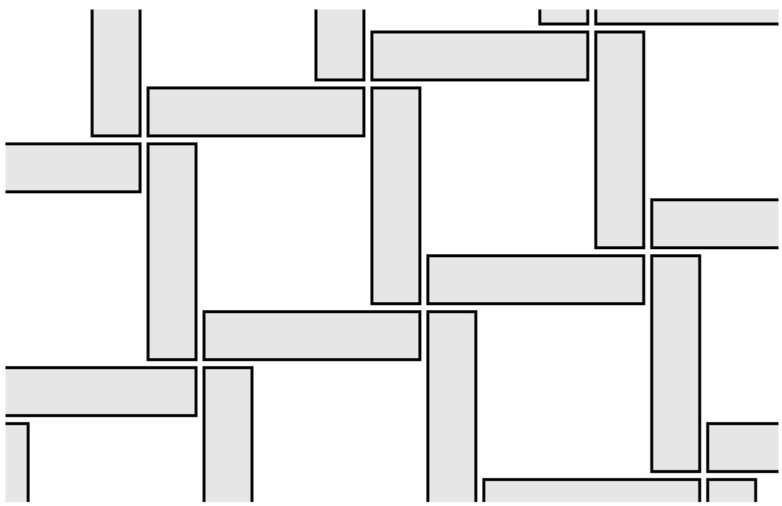

Figure 3: The Conjectured Mistiling for Long Dominoes