Forcing Disjoint Segments in the Plane

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Abstract

Consider a geometric graph given by n points in the plane (in general position) and m line segments, each segment joining a pair of the given points. We show that: if $m \geq 3n+1$ then there are 3 pairwise disjoint segments; if $m \geq 10n+1$ then there are 4 disjoint segments; and if $m \geq c_k n(\log n)^{k-4}$ then there are k disjoint segments.

1 Introduction

A *geometric graph* is a graph drawn in the plane such that its edges are closed line segments and no three vertices are collinear. We let n denote the number of vertices and m the number of edges.

In this paper we consider how many edges are needed to ensure that there are k disjoint edges. This question was raised by Avital and Hanani [2], Kupitz [5] and Perles (see [1]), inter alia.

An old result of Erdős [4] states that if $m \geq n + 1$ for a geometric graph then there are two disjoint edges. Recently Alon and Erdős [1] showed that if $m \geq 6n + 1$ then there are three disjoint edges. O'Donnel and Perles (see [3]) improved this bound to about $3.6n$ edges.

In this paper we show first that $3n + 1$ edges force three disjoint segments. The best lower bound remains $5n/2 - 4$, due to Perles. Our main result is that a linear number of edges, viz. $10n + 1$, force four disjoint edges. We also establish a general upper bound on the number of edges needed to ensure k disjoint segments. It is likely, though, that a linear number of edges suffice.

If the n vertices of the geometric graph are in convex position, then Kupitz [6] and Perles (see [1]) showed that if $m \geq (k-1)n + 1$ then the graph contains k disjoint edges. This bound is tight.

2 Results and Proofs

We say that an edge vu is to the left of edge vw by looking at both edges from vertex v. We say that a vertex v is *pointed* if all the edges incident with it lie in some halfplane with v on the boundary. If v is pointed we may speak of its leftmost and rightmost edges.

The approach to proving results on disjoint edges is to first find a suitable "configuration" and then show that such a configuration must contain the desired number of disjoint edges. To illustrate this approach, we give a proof that $n + 1$ edges imply two disjoint edges. For every vertex mark its rightmost edge, if such an edge exists. An unmarked edge xy remains. Then there is an edge xu to the

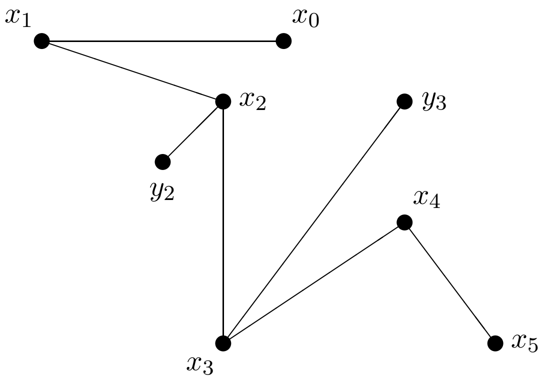

Figure 1: A configuration for Theorem 1

right of xy and an edge yz to the right of yx. These three edges constitute the configuration in this case. As the edges xu and yz are on opposite sides of the line through xy they are disjoint.

The first theorem gives an improved bound for three disjoint edges.

Theorem 1. *If there are n vertices and at least $3n+1$ edges in a geometric graph then there are three disjoint edges.*

Proof: Consider a geometric graph. For each vertex v mark any edge e incident with v for which there does not exist an edge incident with v to the left of e. For a pointed vertex one edge is marked; for a nonpointed vertex no edge is marked.

Now, for each vertex v mark any unmarked edge f incident with v for which there are not two unmarked edges incident with v to the right of f. For a pointed vertex two edges are marked (assuming it has that many); for a nonpointed vertex at most three edges are marked. A total of at most $3n$ edges are marked.

There remains an edge that is unmarked: call it x_2x_3. Then there exist two edges incident with x_3 to the right of x_2x_3, say x_3y_3 and x_3x_4 in order, and two edges incident with x_2 to the right of x_2x_3, say x_2y_2 and x_2x_1. Further there is an edge x_1x_0 to the left of x_1x_2, and an edge x_4x_5 to the left of x_4x_3. If all seven edges were internally disjoint, the configuration might look like the one given in Figure 1.

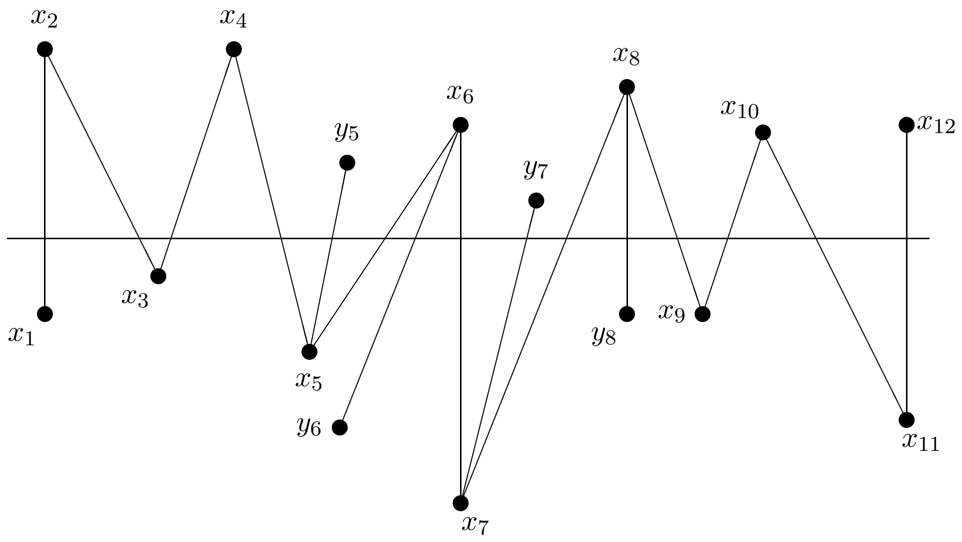

Figure 2: A configuration for Theorem 2

Let l_i denote the line through x_ix_{i+1} ($i = 0, \ldots, 4$). Without loss of generality, the intersection of lines l_1 and l_3 is on the side of l_2 where x_4 is (or at infinity). Then the three edges x_1x_2, x_3y_3 and x_4x_5 are disjoint. (Edges x_1x_2 and x_4x_5 are on opposite sides of l_3.) $\square$

The next theorem gives a linear bound for four disjoint edges.

Theorem 2. *If there are* n *vertices and at least* $10n + 1$ *edges in a geometric graph then there are four disjoint edges.*

Proof: Fix an edge e. If there are $3n + 1$ edges which do not meet e, then the result follows from Theorem 1. Otherwise there are $7n$ edges which meet e. Discard those edges which do not meet e.

Consider the following configuration on sixteen vertices and fifteen edges. The vertices are $x_1, x_2, \ldots, x_{12}$ and $y_5, \ldots, y_8$. The edges are $e_i = x_ix_{i+1}$ ($i = 1, \ldots, 11$), which cross the horizontal edge e in order from left to right, as well as $f_i = x_iy_i$ ($i = 5, \ldots, 8$), which cross e between x_ix_{i-1} and x_ix_{i+1}. If all these edges were internally disjoint it might look like the configuration shown in Figure 2.

It is easy to show that if there are sufficiently many edges then the graph contains such a configuration. To simplify the discussion, we assume for the time being that e is the only edge incident with either of its ends. To obtain the configuration, one may reduce the graph in five steps, so that a remaining edge is the central edge x_6x_7 of the configuration, as follows. For each vertex remove its rightmost edge, then for each vertex remove its leftmost edge, then for each vertex remove its rightmost edge, then for each vertex remove its two leftmost edges, and then for each vertex remove its two rightmost edges. At most $7n$ edges are removed.

Let l_i denote the line through x_ix_{i+1} ($i = 1, \ldots, 11$). We find a suitable foursome in this configuration by an exhaustive case-study. There are two types of foursomes to consider. The first is of the form $e_i, f_{i+2}, f_{i+3}, e_{i+4}$. Note that if neither edge e_i nor edge e_{i+4} intersects the line l_{i+2} then this foursome is disjoint. The second type has the form $e_i, e_{i+2}, e_{i+4}, e_{i+6}$.

Two cases arise. The first is when line l_5 intersects edge e_7. In this case, if edge e_9 meets line l_7 then the edges e_5, e_7, e_9, e_{11} are disjoint; otherwise the edges e_5, f_7, f_8, e_9 are disjoint.

In the second case we may assume by symmetry that the line l_5 misses the edge e_7 and the line l_7 misses the edge e_5. Assume that the lines l_5 and l_7 intersect on the same side of e as is x_6.

Two subcases arise. First assume that edge e_8 meets the line l_6. If the edge e_9 does not meet the line l_7 then edges e_5, f_7, f_8, e_9 are disjoint. If e_9 meets l_7 but not line l_5 then edges e_4, e_6, e_8, e_{10} are disjoint. If e_9 meets l_7 and l_5 then edges e_5, e_7, e_9, e_{11} are disjoint.

So assume that edge e_8 does not meet the line l_6. If the edge e_4 is disjoint from the line l_6, then the edges e_4, f_6, f_7, e_8 are disjoint. If e_4 meets l_6 and edge e_3 meets line l_5 then e_2, e_4, e_6, e_8 are disjoint. If e_4 meets l_6 but edge e_3 does not meet line l_5 then e_3, f_5, f_6, e_7 are disjoint.

The proof where there are other edges incident with the ends of e is similar. Since such edges would be removed in the first two steps of the reduction process, an end of e can only play the role of x_1, x_2, x_{11} or x_{12}. But it is easy to verify that the resultant configuration still contains four disjoint edges. $\square$

It is easy to construct for n odd a geometric graph on n vertices and $7n/2 - 6$ edges without four disjoint edges.

We conclude with a general upper bound on the number of edges needed to guarantee k disjoint edges. Previously only $o(n^2)$ was known.

Theorem 3. *Let $k \geq 5$. If a geometric graph on n vertices has at least $11(3\log n + 1)^{k-4}n$ edges then there are k disjoint edges.*

Proof: We prove by induction on n and $k \geq 5$ that $g(n,k) = 10(3\log n)^{k-4}n + (3\log n + 1)^{k-4}n$ edges suffice. For $k = 4$, $10n + 1$ edges suffice by Theorem 2. The case of n small is easily handled.

Consider a geometric graph with n vertices and m edges. Consider any edge e; say $e = xy$ is horizontal. If at least $g(n, k-1)$ edges do not meet e, then by the inductive hypothesis there are $k - 1$ disjoint edges which are each disjoint from e. So assume otherwise.

Discard the edges that do not meet e, as well as those whose endpoints are x or y. Let f be the remaining edge which has *median* slope. We may assume that less than $g(n, k-1)$ edges do not intersect f.

The lines induced by the two edges e and f divide the plane into four quadrants. (Let the ends of f be in both quadrants.) Each remaining edge has its ends it opposite quadrants. Thus at least $(m - g(n, k-1) - 2n)/2 - g(n, k-1)$ edges connect (vertices in) both pairs of opposite quadrants, by our choice of f. But one pair of quadrants contains at most $n/2$ vertices. Hence we are done provided

$$\left(g(n,k) - 3g(n,k-1) - 2n\right)/2 \geq g(n/2, k).$$

This is easily checked. □

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