Proof of Theorem 5.13

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

By earlier result, we know that T has an eigenvalue, say λ. Let U be the space

$$U = \operatorname{range}(T - \lambda I).$$

We claim that U is invariant under T. For, let $u \in U$. Then we can write Tu as $(T - \lambda I)u + (\lambda I)u$. The first term is in $\operatorname{range}(T - \lambda I)$ by definition; the second term is a multiple of u and hence is in U. It follows that $Tu \in U$.

So we can define operator $T|_U$ as T restricted to U. By induction there is a basis $B' = (u_1, \ldots, u_m)$ of U such that $T|_U$ is represented by some upper-triangular matrix $\mathcal{M}'$.

Now, extend B' to a basis $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ of V. By the same trick as before, we can write Tv_i as $(T - \lambda I)v_i + (\lambda I)v_i$. This means that Tv_i is the sum of λv_i and an element of $\operatorname{range}(T - \lambda I)$ and thus can be written as the sum of λv_i and a linear combination of the u_i.

Hence, T can be represented by the upper triangular matrix:

$$\begin{pmatrix} \mathcal{M}' & \star \\ 0 & \lambda I \end{pmatrix}$$