More on Matrices, Operators, and Traces

Assume $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$; with bases $F = (u_i)$ of U, $G = (v_i)$ of V, and $H = (w_i)$ of W.

Earlier on we defined matrix multiplication so that the following is true:

$$\mathcal{M}(ST, F, H) = \mathcal{M}(S, G, H)\, \mathcal{M}(T, F, G)$$

For, if $\mathcal{M}(S) = (s_{i,j})$ and $\mathcal{M}(T) = (t_{i,j})$, then

$$\begin{aligned} (ST)u_k &= S\left(\textstyle\sum_j t_{j,k} v_j\right) \\ &= \textstyle\sum_j t_{j,k} S v_j \\ &= \textstyle\sum_j t_{j,k} \left(\sum_i s_{i,j} w_i\right) \\ &= \textstyle\sum_i \left(\sum_j s_{i,j} t_{j,k}\right) w_i. \end{aligned}$$

As a consequence, we get

$$\mathcal{M}(I, F, G)^{-1} = \mathcal{M}(I, G, F).$$

Proof: Plug $S = T = I$ and $H = F$ into the above formula. So

$$\mathcal{M}(I, F, F) = \mathcal{M}(I, G, F)\, \mathcal{M}(I, F, G).$$

and clearly $\mathcal{M}(I, F, F)$ is the identity.

As another consequence, we get that

$$\mathcal{M}(T, F, F) = \mathcal{M}(I, F, G)^{-1}\, \mathcal{M}(T, G, G)\, \mathcal{M}(I, F, G).$$

For, we have that $\mathcal{M}(TI, F, G) = \mathcal{M}(T, G, G)\mathcal{M}(I, F, G)$ and $\mathcal{M}(IT, F, G) = \mathcal{M}(I, F, G)\mathcal{M}(T, F, F)$. But the two left-hand-sides are equal. Re-arrange to get desired equation.

Now, define the trace of a matrix as the sum of its diagonal entries. Then, for matrices A and B, $\operatorname{trace}(A + B) = \operatorname{trace} A + \operatorname{trace} B$ straight from the definition of matrix addition. Further,

$$\operatorname{trace}(AB) = \operatorname{trace}(BA).$$

For, $\text{trace}(AB) = \sum_j \sum_k a_{j,k} b_{k,j}$ and $\text{trace}(BA) = \sum_k \sum_j b_{k,j} a_{j,k}$, which are equal.

Further, the big result is:

$$\text{trace}\, \mathcal{M}(T, F) = \text{trace}\, \mathcal{M}(T, G).$$

For,

$$\begin{aligned} \text{trace}\, \mathcal{M}(T, F) &= \text{trace}\left(\mathcal{M}(I, F, G)^{-1}\, \mathcal{M}(T, G)\, \mathcal{M}(I, F, G) \right) \\ &= \text{trace}\left(\mathcal{M}(T, G)\, \mathcal{M}(I, F, G)\, \mathcal{M}(I, F, G)^{-1} \right) \\ &= \text{trace}\, \mathcal{M}(T, G). \end{aligned}$$