Graphs and Determinants

1 The Adjacency Matrix of a Digraph

A ***digraph*** is a collection of vertices and arcs, each arc being an ordered pair of not necessarily distinct vertices. We can define the ***adjacency matrix*** A of a digraph by numbering the vertices, say from 1 up to n, and then putting $a_{ij} = 1$ if there is an arc from i to j, and $a_{ij} = 0$ otherwise. For example,

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

2 Digraphs and Determinants

The determinant of a matrix is defined in terms of permutations. Each permutation is the adjacency matrix of a digraph where each node has exactly one arc going out and exactly one arc going in. We call it a ***CU*** (cycle union).

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3 The Product of Two Permutations

The product of two permutation matrices is another permutation matrix. For, the dot product of two vectors that each have only one 1, is 1 only when these 1's align.

> ***Lemma:*** *The sign of the product is the product of the signs.*

Proof: We can write down the permutation for matrix A as $a(1), a(2), \ldots, a(n)$ where $a(i)$ is the row in column i where the 1 is. The sign of the permutation depends on the parity (oddness/evenness) of how many swaps away from the identity.

So say we are interested in the product AB.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here A has permutation 3241, B has permutation 4123 and AB has permutation 1324. In general, the permutation for AB is $a(b(1)), a(b(2)), \ldots, a(b(n))$. We can swap things to $a(1), a(2), \ldots, a(n)$ using the swaps for permutation B. Then we can get the identity by using the swaps for permutation A. Hence the total number of swaps is the sum of the swaps. And the parity is determined.

4 The Determinant of the Product

Say we have general matrices A and B. Consider an entry (i, j) in the product AB. That entry is a sum of n values, each the product $a_{i,k}$ and $b_{k,j}$. That entry is the label on the arc in the digraph. We can split the arc into n arcs from i to j. Then each CU corresponds to a set of pairs of arcs, one arc from A and one arc from B.

In particular, if we take a CU of A and a CU of B we can identify the corresponding CU of AB. And the weight of the latter is the product of the weights of the former. And we showed above that the sign of the latter is the sign of the former.

Unfortunately however, there are other ways to produce a CU of AB. But, the claim is that their net contribution is zero. Each CU corresponds to a collection of n arcs from A with distinct starts, and n arcs from B with distinct ends. However, the intermediates might not be distinct. For example, we might have one intermediate repeated 3 times.

If we look at all CUs that have that intermediate, there are 3! of them. There is a theorem of algebra that says that the number of even permutations (sign 1) is equal to the number of odd permutations (sign -1). (The simple proof is to form a bijection by swapping the first two entries.) So in our case, the even and odd permutations will have opposite signs and therefore will cancel. The same proof works in general.

Thus the remaining terms of $\det(AB)$ each correspond to a term in $\det A$ and a term in $\det B$. And so, the determinant of AB is the product of the determinants of A and B.

5 Powers of the Adjacency Matrix

The powers of the adjacency matrix counts things. In particular, a ***walk*** in a digraph is a sequence of not necessarily distinct arcs, each arc starting where the other one finishes. Then:

entry i, j in A^k gives the number of walks from i to j of length k.

The proof is a direct induction argument. For example, the number of walks of length 2 is the number of vertices k such that there is an arc from i to k and an arc from k to j. Which is the i, j entry in A^2. For, that is a summation where each term is 0 or 1, and the number of 1s is what we claim.

If we have weights on the edges, then the result is still valid. We redefine the adjacency matrix to have the weights as its entries, and define the weight of a walk as the product of the weights of the arcs. Then if want to know the total sum of weights of i, j paths of given length, that is the entry in the appropriate power.

This result plays a part in the movie *Good Will Hunting*.