A Result on Cages

6 Eigenvalues of Graphs

We consider undirected graphs without loops. A graph is a collection of vertices and edges, each edge joining two neighbors. A graph is r-regular if every vertex has r neighbors. The eigenvalues of a graph are the eigenvalues of its adjacency matrix.

For example: here is a graph which is 2-regular. It has eigenvalues, $2, -1, -1$.

Observation: An r-regular graph has r as an eigenvalue. Proof: It has the all-1 eigenvector.

7 Cages

We are interested in the existence of special graphs. We want an r-regular graph where for some vertex v there is no edge between the neighbors of v, and none of the neighbors have a common neighbor apart from v. Then we must have at least $1 + r + r(r - 1) = r^2 + 1$ distinct vertices. A ***r-supercage*** is an r-regular graph with $r^2 + 1$ vertices with that property for every vertex. (That property is equivalent to no 3- or 4-cycle in the graph.) For example, a 5-cycle is a 2-supercage. And here is the 3-supercage.

Theorem 1 *There is an r-supercage if and only if $r = 2, 3, 7$ or possibly 57.*

PROOF. Assume an r-supercage exists. Let A be its adjacency matrix. Then by the definition, if we consider any pair of vertices, they are either neighbors or they have a unique common neighbor, but not both. Consider what this means for the matrix A^2. Off the diagonal, it has a 1 wherever A has a 0 and vice versa. On the diagonal it has r (just because the graph is r-regular). This means that

$$A^2 + A - (r-1)I = J,$$

where J is the all-1 matrix.

Now let λ be an eigenvalue of A with eigenvector v. There are two possibilities. First, assume the entries of v sum to something nonzero. Then we can normalize v such that its entries sum to 1. Then

$$[\lambda^2 + \lambda - (r-1)]v = J,$$

where 1 is the all-1 vector. It follows that $v = \frac{1}{n}J$, where n is the number of vertices; and that $\lambda = r$ (with multiplicity 1).

The second possibility is that the entries of v sum to zero. Then

$$[\lambda^2 + \lambda - (r-1)]v = 0.$$

And so we have $\lambda^2 + \lambda - (r-1) = 0$. Hence

$$\lambda = \frac{-1 \pm \sqrt{1 + 4(r-1)}}{2}.$$

Now, suppose the $\lambda^+ = (-1 + \sqrt{4r-3})/2$ is an eigenvalue b times. It follows that $\lambda^- = (-1 - \sqrt{4r-3})/2$ is an eigenvalue $n - b - 1 = r^2 - b$ times.

And the trace of A is clearly 0. So we have that

$$b\lambda^+ + (r^2 - b)\lambda^- + r = 0.$$

And thus that

$$2b - r^2 = \frac{r(r-2)}{\sqrt{4r-3}}.$$

The right-hand side must therefore be an integer. If the numerator is 0, which happens when $r = 2$, then we are okay. Otherwise, we certainly need that $\sqrt{4r-3}$ be rational, which only happens if it is an integer, say m.

Substitute this into the RHS, and we get

$$\frac{m^4 - 2m^2 - 15}{16m}$$

Thus, m must be a divisor of 15, which forces $m \in \{1, 3, 5, 15\}$. Thus $r \in \{1, 3, 7, 57\}$. It can be checked that in each case of r, b is a whole number.

r-supercages have been constructed for $r = 2, 3, 7$, but the existence for $r = 57$ is unsolved.