Summary of Axler Chapter 7: Part 2

(Theorems 7.27, 7.36, and 7.37 with proof. Theorems 7.41 and 7.46 without.)

An operator T is positive if T is self-adjoint and $\langle Tv, v\rangle \geq 0$ for all v. For example, an orthogonal projection is positive. Equivalent are: (a) T is positive, (b) T is self-adjoint and all eigenvalues are nonnegative; (d) T has a self-adjoint square root; (e) $T = S^*S$ for some operator S.

An operator S is called an isometry if $||Sv|| = ||v||$ for all v. For example, a rotation is an isometry. Equivalent are: (a) S is an isometry, (b) $S^*S = SS^* = I$. So an isometry is normal. Over a complex vector space, S is an isometry iff there is an orthonormal basis of V consisting of eigenvectors of S with all $|\lambda_i| = 1$.

The Polar Decomposition Theorem states that for all $T \in \mathcal{L}(V)$ there is an isometry $S \in \mathcal{L}(V)$ such that

$$T = S\sqrt{T^*T}.$$

This is analogous to writing a complex number as rotation times radius.

The singular values of T are the eigenvalues of $\sqrt{T^*T}$ with the eigenvalue λ repeated $\dim \operatorname{null}(\sqrt{T^*T} - \lambda I)$ times. The singular values are all nonnegative.

The Singular Value Decomposition Theorem states that for all $T \in \mathcal{L}(V)$, there exists an orthornormal basis (e_i) and specific elements (f_i) such that for all v

$$Tv = \sum s_i \langle v, e_i\rangle f_i,$$

where (s_i) are the singular values.