# Summary of Axler Chapter 6b

By Gram–Schmidt: If $T$ has an upper-triangular matrix for some basis, then $T$ has an upper-triangular matrix with respect to some orthornormal basis of $V$.

The orthogonal complement of set $U$ is the set of vectors that are orthogonal to every vector in $U$, written $U^\perp$. $U^\perp$ is always a subspace. Theorem: If $U$ is a subspace of $V$, then $V = U \oplus U^\perp$. (To show $U + U^\perp$ spans $V$, start with an orthonormal basis of $U$.) Also, $U = (U^\perp)^\perp$.

Given subspace $U$ of $V$, the orthogonal projection operator $P_U\colon V \to U$ is defined by mapping $v$ to $u$ where $v = u + w$ with $u \in U$ and $w \in U^\perp$. Theorem: $||v - P_U v|| \leq ||v - u||$ for every $u \in U$. That is, $P_U v$ is the "closest" element of $U$ to $v$.

A linear functional is a linear map whose codomain is the scalars $\mathbf{F}$. Theorem: If $\varphi$ is a linear functional, then there is a unique vector $v$ such that $\varphi(u) = \langle u, v\rangle$ for all $u$. (For the existence, start with an orthonormal basis of $V$; then $v = \sum_i \overline{\varphi(e_i)}\, e_i$.)

Let $T \in \mathcal{L}(V, W)$. Then the adjoint of $T$, denoted $T^*$, is in $\mathcal{L}(W, V)$, and is defined so that $\langle Tv, w\rangle = \langle v, T^* w\rangle$ for all $v, w$. There is a connection between the null space, range, orthogonal complement, and adjoint. For example, $\text{null}(T^*) = (\text{range}\, T)^\perp$. The matrix of the adjoint $T^*$ is given by the conjugate transpose of the matrix of $T$ (provided $T$ was given in terms of an orthonormal basis).