## Summary of Axler Chapter 6a

Let $\mathbf{F}$ be either $\mathbf{R}$ or $\mathbf{C}$. An inner product is a function that maps ordered pairs of elements to the underlying field, denoted $\langle u, v\rangle$, such that (i) $\langle v, v\rangle \geq 0$ with equality only for 0 vector; (ii) linearity under addition; (iii) homogeneity in the first slot; and (iv) conjugate symmetry.

Examples include: the standard dot product for $\mathbf{R}^n$; the conjugate dot product for $\mathbf{C}^n$; and for $\mathcal{P}(\mathbf{R})$ we can define $\langle p, q\rangle = \displaystyle\int_0^1 p(x)\,\overline{q(x)}\,dx$.

Two vectors are orthogonal iff their inner product is zero. The norm of a vector is $||v|| = \sqrt{\langle v, v\rangle}$.

Pythagoras' theorem: If $u$ and $v$ are orthogonal, then

$$||u+v||^2 = ||u||^2 + ||v||^2.$$

From this follows: Parallelogram Equality: $||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2)$.

Cauchy-Schwarz:

$$|\langle u, v\rangle| \leq ||u||\,||v||.$$

From this follows: Triangle inequality: $||u+v|| \leq ||u|| + ||v||$.

An orthonormal set is a set where the vectors are mutually orthogonal and all have norm 1. If $(e_i)$ is an orthonormal basis, then the coefficients expressing vector $v$ in terms of the basis are given by the inner product: $v = \sum_i \langle v, e_i\rangle e_i$.

A nonzero vector is normalized by dividing by its norm. The Gram-Schmidt process converts a linearly independent set $(v_i)$ into an orthonormal set $(e_i)$ with the same span. For increasing $j$, each $e_j$ is the normalized version of

$$v_j - \sum_{i<j} \langle v_j, e_i\rangle e_i$$