Axler Chapter 5 (part 2)

When constructing the matrix of an operator (which maps a space to itself), only one basis is used. A matrix is upper triangular if all the entries below the diagonal are 0. The matrix is diagonal if all the entries off the diagonal are 0.

If V is a complex vector space, then any operator T has an upper-triangular matrix with respect to some basis. The proof is by induction on the dimension of V, starting with the fact that T must have an eigenvalue.

If T has an upper-triangular matrix $\mathcal{M}$ with respect to some basis, then (a) the eigenvalues of T are precisely the diagonal entries of $\mathcal{M}$, and (b) T is invertible iff all the diagonal entries of $\mathcal{M}$ are nonzero.

An operator has a diagonal matrix if and only if it has a basis consisting of eigenvectors. In particular, if a vector space has $\dim V$ distinct eigenvalues values, then it has a diagonal matrix. If $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues of T and V has a basis of eigenvectors, then $V = \text{null}(T - \lambda_1 I) \oplus \cdots \oplus \text{null}(T - \lambda_m I)$.

Every operator on a finite-dimensional nonzero real vector space has an invariant subspace of dimension 1 or 2. The proof again uses the fundamental theorem of algebra.

Every operator on an odd-dimensional real vector space has an eigenvalue. In order to use induction, the proof uses the projection operator. If $V = U \oplus W$, then the projection operator $P_{U,W}$ applied to vector v keeps the u part of the sum $v = u + w$.