Axler Chapter 5 (part 1)

A subspace U is invariant under an operator T if $Tu \in U$ for all $u \in U$. For example, the null space of T is invariant under T.

If U has dimension 1, then this means that u is mapped to a multiple of u. An eigenvalue of T is a scalar λ such that there is a nonzero u such that

$$Tu = \lambda u \qquad \text{or equivalently} \qquad (T - \lambda I)u = 0.$$

The u is an eigenvector. For example, if $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(w, z) = (-z, w)$ (rotation through 90 deg), then the eigenvalues are $\pm i$ with eigenvectors $(1, -i)$ and $(1, i)$.

Eigenvectors from distinct eigenvalues are linearly independent. Thus the number of eigenvalues is at most the dimension n of the space.

Every operator on a finite-dimensional nonzero complex vector space has an eigenvalue. The proof idea is to observe that $(v, Tv, T^2v, \ldots, T^nv)$ is linearly dependent, and thus one gets a polynomial condition; by the fundamental theorem of algebra this condition can be factored into linear form $c(T - \lambda_1 I) \ldots (T - \lambda_m I)v = 0$, and so one of the linear forms is not injective.