

Summary of Axler Chapter 4: Polynomials

A root of a polynomial is a value that makes the polynomial zero. The value λ is a root of $p(z)$ iff $p(z) = (z - \lambda)q(z)$ for a polynomial $q(z)$ with smaller degree. It follows by induction that a polynomial of degree m has at most m roots.

The Fundamental Theorem of Algebra says that every nonconstant polynomial with complex coefficients has a root. And so by induction, every such polynomial of degree m can be written as

$$p(z) = c(z - \lambda_1) \dots (z - \lambda_m)$$

for some $c, \lambda_i \in \mathbf{C}$. Furthermore that formulation is unique up to the order of the λ_i .

Note that if λ is a root of a polynomial with real coefficients, then so is its complex conjugate $\bar{\lambda}$. A real polynomial $x^2 + \alpha x + \beta$ has real roots iff $\alpha^2 \geq 4\beta$. A real polynomial has a unique factorization

$$p(x) = c \prod_i (x - \lambda_i) \prod_j (x^2 + \alpha_j x + \beta_j)$$