Summary of Axler Chapter 3 (part 1)

A linear map T from one vector space V to another W is a map that preserves the sum and the scalar product: $T(u+v) = Tu + Tv$ and $T(av) = a(Tv)$. For example, the zero map 0 maps everything to zero; the identity map I maps everything to itself. Other examples include differentiation of polynomials and backward shift of sequences.

A linear map is completely determined by what it does to a basis of V. The set $\mathcal{L}(V, W)$ of all linear maps is itself a vector space. Multiplication of maps means composition: $(ST)(v) = S(Tv)$. It has some nice properties.

The null space or kernel of a map, denoted $\operatorname{null} T$, is the set of all elements that get mapped to 0. It is a subspace of V. The range of a map is the set of all elements that get mapped to. It is a subspace of W. A map is injective (one-to-one) iff the null space is $\{0\}$. A map is surjective (onto) iff the range is all of W.

If T is a linear map from (finite-dimensional) V to W, then

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

The proof idea is to extend a basis of the null space of T to a basis of V and show that the image of the extra vectors is a basis of the range.

Consequently, for example, if V has bigger dimension then W, then no map is injective; if W has bigger dimension then V, then no map is surjective. The solution set of a system of homogeneous linear equations is a vector space.