Summary of Axler Chapter 2

A linear combination of vectors v_i in V is $\sum_i a_i v_i$ for some scalars a_i. The span of $(v_1, \dots v_m)$ is the set of all linear combinations of them. The span is a subspace. For example, the span of $(1, z, z^2, \dots, z^m)$ is the space $\mathcal{P}_m(\mathbf{F})$ of all polynomials of degree at most m. A vector space is called finite-dimensional if it is spanned by a finite set. The span of the empty set is $\{0\}$.

A list of vectors v_i in V is linearly independent if $\sum_i a_i v_i = 0$ means all a_i are 0. Otherwise it is linearly dependent. For example, $(1, z, z^2, \dots, z^m)$ is linearly independent.

We prove that if a set is linearly dependent, then one can always remove one vector and the span of the set is unchanged. So for example if a finite set is linearly dependent, then it contains a subset that is linearly independent but still has the same span. Conversely, if a vector space is finite-dimensional, then every linearly dependent set can be extended to a spanning set while remaining linearly independent. The proofs are constructive.

A basis of a vector space V is a set that is linearly independent and spans V. The key result is that any two bases have the same size. Thus, the dimension of a vector space is the size of a basis of the space. Note that if B is a basis, every element of V can be written uniquely as a linear combination of elements of B. If V has dimension n and we have a set of size n that either spans V or is linearly independent, then it must be a basis of V.

The dimension of a subspace U is at most that of V. Note that $\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$. So, if $V = U_1 + U_2$ and $\dim U_1 + \dim U_2 = V$, then this is a direct sum.