## Axler Chapter 10 part 1

Recall that a matrix $A$ is invertible (or nonsingular) if there is a matrix $B$ such that $AB = BA = I$. We write $A^{-1}$ for the inverse if it exists.

Recall that given $T \in \mathcal{L}(V, W)$ with basis $B$ of $V$ and $C$ of $W$, then we write $\mathcal{M}(T)$ or more formally $\mathcal{M}(T, B, C)$ as the matrix of $T$ with respect to those bases. We show that if $V = W$ then $\mathcal{M}(I, B, C)^{-1} = \mathcal{M}(I, C, B)$ and in general

$$\mathcal{M}(T, B) = \mathcal{M}(I, B, C)^{-1}\,\mathcal{M}(T, C)\,\mathcal{M}(I, B, C).$$

The characteristic polynomial of operator $T$ over $\mathbf{C}$ is defined as

$$(z - \lambda_1)^{d_1} \dots (z - \lambda_m)^{d_m},$$

where $\lambda_i$ are the eigenvalues with multiplicity $d_i$.

The trace of operator $T$ over $\mathbf{C}$ is the sum of the full set of eigenvalues. Equivalently, it is the coefficient of $z^{n-1}$ in the characteristic polynomial. The trace of a square matrix is the sum of the diagonal entries. For any square matrices $A$ and $B$, $\operatorname{trace}(AB) = \operatorname{trace}(BA)$ and $\operatorname{trace}(A+B) = \operatorname{trace} A + \operatorname{trace} B$. We show that for $T \in \mathcal{L}(V)$ that $\operatorname{trace} T = \operatorname{trace}(\mathcal{M}(T))$.

The determinant of an operator $T$ over $\mathbf{C}$ is the product of the full set of eigenvalues. Equivalently, it is the constant term in the characteristic polynomial, and the characteristic polynomial of $T$ is the determinant of the operator $zI - T$. An operator is invertible if and only if its determinant is nonzero.

We define the determinant of matrix $A$ as

$$\det A = \sum (\operatorname{sign}(m_1, \dots, m_n))\, a_{m_1,1} \cdots a_{m_n,n}$$

where the sum is over all permutations $(m_1, \dots, m_n)$. Interchanging two columns of a matrix changes the sign of the determinant. For matrices/operators $A$ and $B$

$$\det(AB) = \det A \det B.$$

Finally we show that we show that for $T \in \mathcal{L}(V)$ that $\det T = \det(\mathcal{M}(T))$.