

Summary of Axler Chapter 1 (as expanded)

A field is a collection of elements on which addition and multiplication are defined such that they obey the usual laws of arithmetic that we associate with the real numbers (such as associative, commutative, and distributive laws and the existence of identities and inverses).

The complex numbers $a + bi$ have a real and imaginary part. $i^2 = -1$. We denote \mathbf{R} for the field of reals, and \mathbf{C} for the field of complex numbers. Another field is \mathbf{Z}_p , where p is a prime: this contains the numbers $\{0, 1, \dots, p-1\}$ with addition and multiplication defined modulo p . The existence of multiplicative inverses follows from Euclid's algorithm.

Given a field \mathbf{F} , a vector space V is a set with two operations, addition and scalar multiplication (a scalar is a member of \mathbf{F}) such that addition is commutative, associative, has an identity (written 0) and has inverses; scalar multiplication respects the multiplicative identity of \mathbf{F} (written 1); and the distributive law holds.

Examples of vector spaces:

\mathbf{F}^n denotes the set of ordered n -tuples from \mathbf{F} with component-wise addition and scalar multiplication.

\mathbf{F}^∞ denotes the set of sequences from \mathbf{F} .

$\mathcal{P}(\mathbf{F})$ denotes the set of polynomials with coefficients in \mathbf{F} .

The complex numbers are themselves a vector space over the reals.

We note that: a vector space has a unique additive identity; each element has a unique additive inverse; $0v = a0 = 0$; $(-1)v = -v$.

A subspace U of vector space V is a nonempty subset of V that is closed under addition and scalar multiplication. Both $\{0\}$ and V are subspaces. For example, the subspaces of \mathbf{R}^2 (the plane) are $\{0\}$, \mathbf{R}^2 , and all lines in \mathbf{R}^2 through the origin.

The intersection of two subspaces is a subspace. The sum of two subspaces U and V , written $U + V$, is the set of sums of elements one from U and one from V . If $U \cap V = \{0\}$, then this is the direct sum $U \oplus V$ and every element in $U \oplus V$ can be written uniquely as a sum of an element from U and an element from V .