The Cayley–Hamilton Theorem

Let T be any operator over complex finite-dimensional space. Then T (and hence its matrix A) satisfies its characteristic polynomial.

The characteristic polynomial of T is $q(z) = \prod_j (z - \lambda_j)$ where the λ_j are the not necessarily distinct eigenvalues. We need to show $q(T) = 0$. This is equivalent to $q(T)\, v_i = 0$ for all $v_i \in B$ for some basis.

Choose B such that T has an upper-triangular matrix C. Then C has the eigenvalues λ_i on the diagonal. Define $q_i(z) = \prod_{j \le i} (z - \lambda_j)$. Then we prove that

$q_i(T)\, v_j = 0$ *for all* $j \le i$.

The proof is by induction on i.

For example, $Tv_1 = \lambda_1 v_1$. That is, $(T - \lambda_1 I)v_1 = 0$. And $T - \lambda_1 I = q_1(T)$.

In general, assume the inductive hypothesis. Now,

$$T\, v_i = \sum_{j<i} c_{ji} v_j \, + \lambda_i v_i \, ,$$

and thus $(T - \lambda_i I)v_i$ is a linear combination of $v_1, \ldots, v_{i-1}$. By the inductive hypothesis, $q_{i-1}(T)\, v_j = 0$ for all $j \le i - 1$. Thus $q_{i-1}(T)(T - \lambda_i I)\, v_i = 0$. But $q_{i-1}(T)(T - \lambda_i I) = q_i(T)$.

At the same time, $q_i(T) = q_{i-1}(T)\,(T - \lambda_i I) = (T - \lambda_i I)\, q_{i-1}(T)$ (even though operators don't commute, two polynomials in the same operator do). So it follows that $q_i(T)v_j = (T - \lambda_i I)\, 0 = 0$ for all $j < i$.

Hence $q_i(T)\, v_j = 0$ for all $j \le i$. And so $q(T)\, v_i = 0$ for all i. As required.