Proof on Wikipedia of Cauchy-Schwarz

As the inequality is trivially true in the case $y = 0$, we may assume $\langle y, y\rangle$ is nonzero. Let be a complex number. Then,

$$0 \leq \|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y\rangle = \langle x, x\rangle - \bar{\lambda}\langle x, y\rangle - \lambda\langle y, x\rangle + |\lambda|^2\langle y, y\rangle.$$

The above expression is valid for any complex . Choosing

$$\lambda = \langle x, y\rangle \cdot \langle y, y\rangle^{-1}$$

we obtain

$$0 \leq \langle x, x\rangle - |\langle x, y\rangle|^2 \cdot \langle y, y\rangle^{-1}$$

which is true if and only if

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle$$

or equivalently:

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| \,,$$

which is the CauchySchwarz inequality.