Graceful Graphs with Arbitrarily High Chromatic Number

Allen J. Schwenk
Western Michigan University

G has order *n* and size *m*.

G is graceful if we can select distinct vertex labels from $\{1, 2, \cdots, m\}$,
and then set edge labels to be

$$\ell(v_i v_j) = \left|\ell(v_i) - \ell(v_j)\right|$$

so that each edge label $\{1, 2, \cdots, m\}$ appears exactly once.

K_n is graceful if and only if $1 \leq n \leq 4$.

Also, Chartrand, Hevia, and Oellermann, *The Chromatic Number of a Factorization of a Graph*, *Bull. Inst. Math & Applic.*, **20**(1997), 33-56 observed

$K_2 + C_5$ is graceful and $\chi(K_2 + C_5) = 5$
and they didn't know any
graceful *G* with larger $\chi(G)$

Chartrand and Lesniak reported this as a conjecture in
Graphs & Digraphs, 4th edition, p266,

Conjecture:
Graceful graphs with arbitrarily large chromatic numbers do not exist.
Here is a construction for a graceful graph with $\chi(G_t) = t$ for each *t*.

$$V(G_t) = \{1,2,\cdots 2^{t-1}\} \cup (2^t\}$$

$$E(G_t) = \{ij : j = 2^k \quad and \quad 2i \le j\}$$

$$\text{order}(G_t) = 2^{t-1} + 1$$

$$\text{size}(G_t) = 2^t - 1$$

labels $\ell(i) = i - 1$

Example: For $t = 7$

Each vertex of the form 2^k
is joined to every vertex above it

1

2

3 4

5 6 7 8

9 10 11 . . . 15 16

17 18 19 . . . 31 32

64

The family G_t shows we can attain every

$\chi(G_t) = t$ but it uses order $(G_t) = 2^{t-1} + 1.$

I am told that this construction was given in a paper

D. Acharya and S. Arumugam,
Embedding and NP-complete problems for graceful graphs, in
Labellings of Discrete Strructures and Applications, editors: D. Acharya, S. Arumugam, and A. Rosa

This order $(G_t) = 2^{t-1} + 1$ seems excessive.

How small can the order be to attain

$\chi(G_t) = t$?

For a nongraceful graph,

the gracefulness of G,
grac(G) = smallest k

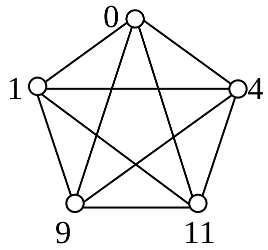
such that we can label $V(G)$ with distinct labels taken from $\{0, 1, 2, \cdots, k\}$ so that the induced edge labels are distinct.

By a VERY time intensive computer search I found

n	$grac(K_n)$
5	11
6	17
7	25
8	34
9	44
10	55
11	72
12	85
13	106

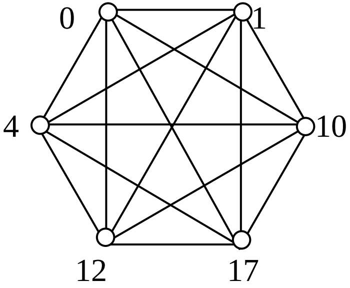
By adding a few vertices and edges to each of these complete graphs, we can get a graph with $\chi(G) = t$ and order quite small.
But I cannot say with certainty that it is the smallest order

$grac(K_5) = 11$



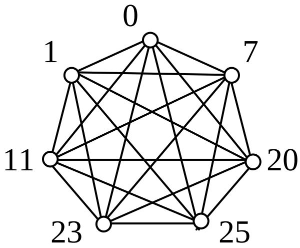
We can make this graceful by adding one vertex labeled 6

$grac(K_6) = 17$



We can make this graceful by adding one vertex labeled 15 and two edges.

$grac(K_7) = 25$



We can make this graceful by adding two vertices labeled 3 and 22 and four edges,

3-11 3-20 1-22 7-22

Continuing in this manner, we find

order of *G*

n	$grac(K_n)$	added verts	added edges	with $\chi(G) = n$ & G graceful
5	11	6	one	6
6	17	15	two	7
7	25	3, 22	four	9
8	34	20,28,30	six	11
9	44	4, 33, 43	eight	12
10	55	5,38,45,52	ten	14
11	72	2, 8, 10, 16, 36, 66	seventeen	17
12	85	5, 11, 18, 72, 83, 84	nineteen	18
13	106	4, 7, 23, 27 54, 82, 90, 105	twentyeight	21

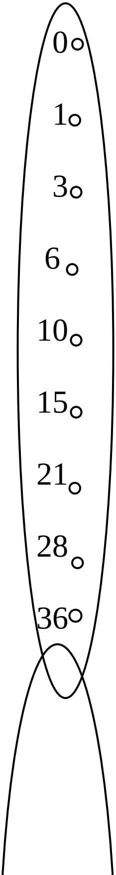
How fast is the order growing?

Maybe faster than linear

Conjecture

The smallest order of G with G graceful and $\chi(G) = n$ grows faster than $c_1 n$ but slower than $c_2 n^2$.

Smaller construction for K_n illustrated for K_9



1							
	3						
2		6					
	5		10				
3		9		15			
	7		14		21		
4		12		20		28	
	9		18		27		36
5		15		25		35	
	11		22		33		
6		18		30			
	13		26				
7		21					
	15						
8							

To remove any possible repeats, add $\Delta =$ to each interval

Construction for K_n illustrated for K_9

0 ∘									
	13								
13 ∘		27							
	14		42						
27 ∘		29		58					
	15		45		75				
42 ∘		31		62		93			
	16		48		80		112		
58 ∘		33		66		99		132	
	17		51		85		119		
75 ∘		35		70		105			
	18		54		90				
93 ∘		37		74					
	19		57						
112 ∘		39							
	20								
132 ∘									

To remove repeats

add $\Delta = 12$ to each interval

Insert vertices 114, 115, …, 131

Add edges as needed to get G_9

G_9 has order $9 + 9 + \Delta = 30$ and $\chi = 9$

What is the smallest Δ that guarantees no repeated values?

For $n = 2k$, we need

$$k\Delta + \binom{k+1}{2} = 1 + (k-1)\Delta + (k-1)n - \binom{k}{2}$$

$$\Delta = 1 + (k-1)2k - \binom{k}{2} - \binom{k+1}{2}$$

$$\Delta = k^2 - 2k + 1$$

The farthest right interval has length $\Delta + n - 1 = k^2$

Insert $k^2 - 2$ additional vertices, and as many edges as needed to make the resulting graph graceful

$\chi = n = 2k$, and order is $k^2 + 2k - 2 = \dfrac{n^2 + 4n - 8}{4}$

For $n = 2k - 1$, we need

$$k\Delta + \binom{k+1}{2} = 1 + (k-1)\Delta + (k-1)n - \binom{k}{2}$$

$$\Delta = 1 + (k-1)(2k-1) - \binom{k}{2} - \binom{k+1}{2}$$

$$\Delta = k^2 - 3k + 2$$

The farthest right interval has length $\Delta + n - 1 = k^2 - k$

Insert $k^2 - k - 2$ additional vertices, and as many edges as needed to make it graceful

$\chi = n = 2k - 1,$ and order is

$$n + \Delta + n - 1 - 2 = k^2 + k - 3 = \frac{n^2 + 4n - 9}{4} = \left\lfloor \frac{n^2 + 4n - 8}{4} \right\rfloor$$

How fast is the order growing?

Maybe faster than linear

Conjecture
The smallest order of G with G graceful and $\chi(G) = n$ grows faster than $c_1 n$ but slower than $c_2 n^2$.

We just showed that the smallest order with

$$\chi(G) = n$$

is not more than
$$\left\lfloor \frac{n^2 + 4n - 8}{4} \right\rfloor$$

Question: Can we find an infinite family that grows slower than $O(n^2)$?