Packing Degree Sequences

Stephen G. Hartke

Department of Mathematics

University of Nebraska–Lincoln

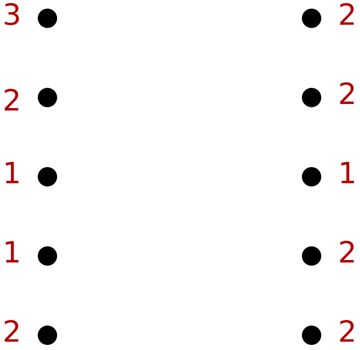
www.math.unl.edu/∼shartke2

hartke@math.unl.edu

Joint work with Art Busch, Mike Ferrara, Mike Jacobson,

Hemanshu Kaul, Tyler Seacrest, and Doug West

Shipping Problem

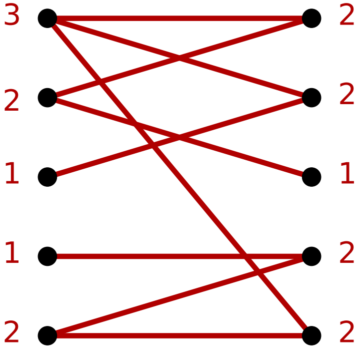


Each factory produces f_i of a good, and each store has demand r_j.

Each factory can only ship one unit to any given store.

Can a shipping plan be made?

Shipping Problem



Each factory produces f_i of a good, and each store has demand r_j.

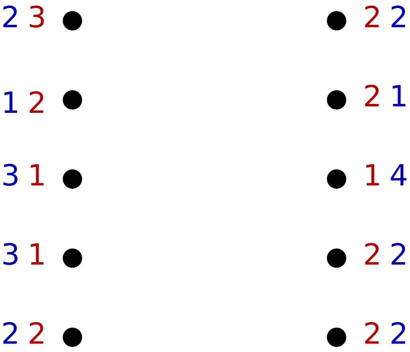
Each factory can only ship one unit to any given store.

Can a shipping plan be made?

Can be solved efficiently as a network flow.

Equivalent to finding a bipartite graph with these degrees.

Shipping Problem

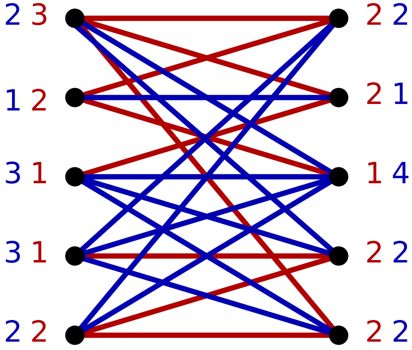


Produce f_i' of good 1 and f_i'' of good 2 with corresponding demands r_j' and r_j''.

Each factory can only ship one unit to any given store.

Can a shipping plan be made?

Shipping Problem



Produce f_i' of good 1 and f_i'' of good 2 with corresponding demands r_j' and r_j''.

Each factory can only ship one unit to any given store.

Can a shipping plan be made?

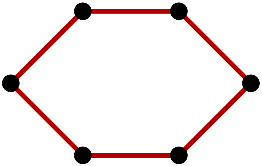
Can be modeled as a multi-commodity flow. Complexity?

Equivalent to packing bipartite graphs with these degrees.

Graphic Sequences

A nonnegative integer sequence π is graphic
if π is the degree sequence of some (simple) graph G.

$$\pi = (2, 2, 2, 2, 2, 2)$$



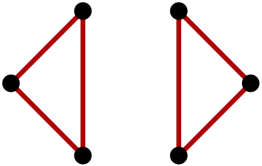
In this case, we say G realizes π and write

$$G = G(\pi) \text{ or } \pi = \pi(G).$$

Graphic Sequences

A nonnegative integer sequence π is graphic
if π is the degree sequence of some (simple) graph G.

$$\pi = (2, 2, 2, 2, 2, 2)$$

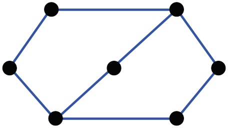


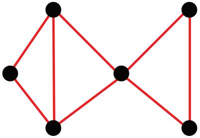
In this case, we say G realizes π and write

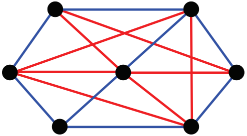
$$G = G(\pi) \text{ or } \pi = \pi(G).$$

Graph Packings

Graphs G_1 and G_2 pack if there is a simple graph G consisting of edge-disjoint copies of G_1 and G_2 or, alternatively, if $G_1 \subseteq \overline{G_2}$.







Degree Sequence Packings

Let $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ be graphic seqs.

π_1 and π_2 pack if there exist $G_1 = G(\pi_1)$ and $G_2 = G(\pi_2)$

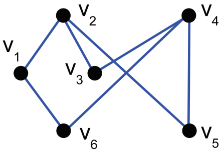
with

$$V(G_1) = V(G_2) = \{v_1, \ldots, v_n\},$$

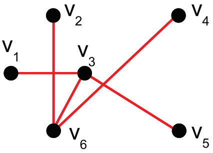
$$E(G_1) \cap E(G_2) = \emptyset,$$

$$\deg_{G_1}(v_i) = d_i^{(1)} \text{ and } \deg_{G_2}(v_i) = d_i^{(2)}.$$

An Example

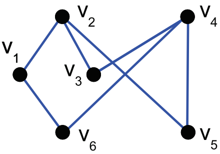


(2,3,2,2,3,2)

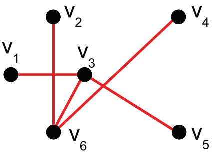


(1,1,3,3,1,1)

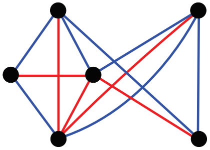
An Example



(2,3,2,2,3,2)

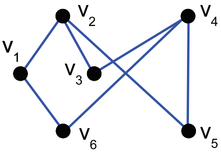


(1,1,3,3,1,1)

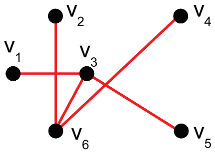


(3,4,5,5,4,3)

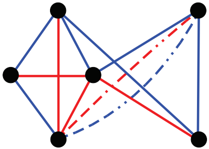
An Example



(2,3,2,2,3,2)

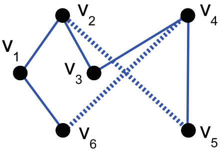


(1,1,3,3,1,1)

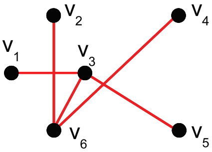


(3,4,5,5,4,3)

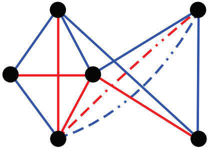
An Example



(2,3,2,2,3,2)

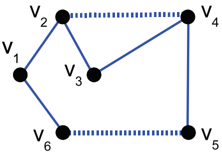


(1,1,3,3,1,1)

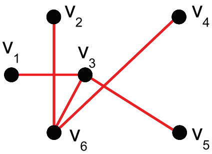


(3,4,5,5,4,3)

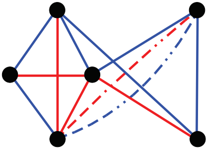
An Example



(2,3,2,2,3,2)

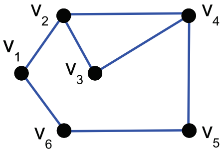


(1,1,3,3,1,1)

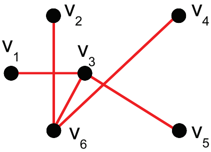


(3,4,5,5,4,3)

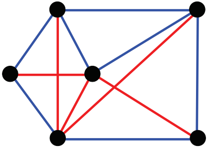
An Example



(2,3,2,2,3,2)



(1,1,3,3,1,1)



(3,4,5,5,4,3)

Permutation of Vertices?

When packing graphs, we permit permuting the vertices to make G_1 and G_2 “fit together”.

When packing sequences, we do **not** permit the sequences π_1 and π_2 to be permuted.

Permutation of Vertices?

When packing graphs, we permit permuting the vertices to make G_1 and G_2 “fit together”.

When packing sequences, we do **not** permit the sequences π_1 and π_2 to be permuted.

Let ρ_1 be π_1 in nonincreasing order
and ρ_2 be π_2 in nondecreasing order.

Thm. If π_1 and π_2 pack, then ρ_1 and ρ_2 pack.

$$\pi_1 = (2,3,2,2,3,2) \qquad \rho_1 = (3,3,2,2,2,2)$$
$$\pi_2 = (1,1,3,3,1,1) \qquad \rho_2 = (1,1,1,1,3,3)$$

A Necessary Condition

Let $\pi_1 + \pi_2$ denote the term-wise sum of π_1 and π_2.

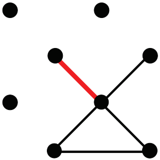
If π_1 and π_2 pack, then $\pi_1 + \pi_2$ clearly must be graphic.

A Necessary Condition

Let $\pi_1 + \pi_2$ denote the term-wise sum of π_1 and π_2.

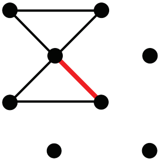
If π_1 and π_2 pack, then $\pi_1 + \pi_2$ clearly must be graphic.

However, this condition is in general not sufficient:



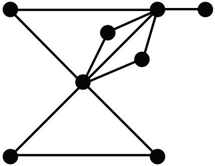
G_1

(0,0,1,1,0,4,2,2)



G_2

(2,2,4,0,2,2,0,0)



G_{1+2}

(2,2,5,1,2,6,2,2)

A Graph Packing Result

Thm. [Sauer–Spencer, 1978]

Let G_1 and G_2 be n-vertex graphs with max degrees Δ_1 and Δ_2.

If $\Delta_1\Delta_2 < n/2$, then G_1 and G_2 pack.

A Sauer-Spencer Type Result

Thm.

Let $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ be graphic seqs.

If $\Delta = \max\{d_i^{(1)} + d_i^{(2)}\}$ and $\delta = \min\{d_i^{(1)} + d_i^{(2)}\}$
are such that $\delta > 0$ and $\Delta < \sqrt{2\delta n} - (\delta - 1)$,

then π_1 and π_2 pack, provided that $\pi_1 + \pi_2$ is graphic.

This result is sharp.

Sharpness

Let m be an even integer, $n = 2m^2$.

$$\pi_1 = (m, m, 2m^{m-1}, 0^{m-1}, 1^{m^2-m}, 0^{m^2-m})$$

$$\pi_2 = (m, m, 0^{m-1}, 2m^{m-1}, 0^{m^2-m}, 1^{m^2-m})$$

$$\pi_1 + \pi_2 = (2m^{2m}, 1^{m^2-m}),$$

$$\Delta = 2m \quad \delta = 1 \quad n = \sqrt{2\delta\Delta} - (\delta - 1)$$

Sharpness

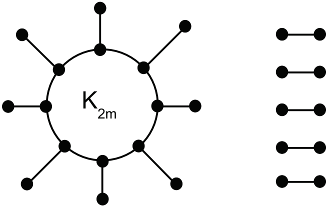
Let m be an even integer, $n = 2m^2$.

$$\pi_1 = (m, m, 2m^{m-1}, 0^{m-1}, 1^{m^2-m}, 0^{m^2-m})$$

$$\pi_2 = (m, m, 0^{m-1}, 2m^{m-1}, 0^{m^2-m}, 1^{m^2-m})$$

$$\pi_1 + \pi_2 = (2m^{2m}, 1^{m^2-m}), \quad \text{is graphic}$$

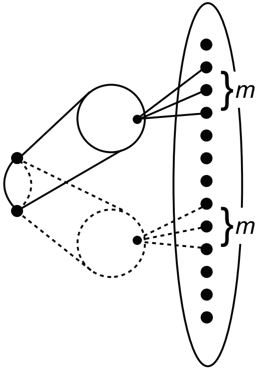
$$\Delta = 2m \quad \delta = 1 \quad n = \sqrt{2\delta\Delta} - (\delta - 1)$$



Sharpness

$$\pi_1 = (m, m, 2m^{m-1}, 0^{m-1}, 1^{m^2-m}, 0^{m^2-m})$$

$$\pi_2 = (m, m, 0^{m-1}, 2m^{m-1}, 0^{m^2-m}, 1^{m^2-m})$$



Kundu's k-Factor Theorem

Thm. [Kundu, 1973]

Let k be a positive integer, and let π_1 and π_2 be graphic sequences such that each term in π_2 is k or $k-1$.

Then π_1 and π_2 pack if and only if $\pi_1 + \pi_2$ is graphic.

Extending Kundu’s Theorem

Rao and Rao showed the following while attempting to prove the (then) k-factor conjecture.

Lemma [A.R. Rao and S.B. Rao, 1972]

Let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$.

Then for any nonnegative integer $r \leq k$ such that rn is even, $\pi - r = (d_1 - r, \ldots, d_n - r)$ is also graphic.

Therefore, if some realization of G has a k-factor,
then there is also a realization that contains an r-factor for any (feasible) $r < k$.

A Conjecture

We conjecture that Kundu's Theorem can be strengthened in the following manner.

Conj. Let $k > 0$ and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic.

Then for any $k_1, \ldots, k_t$ such that nk_i is even for all i and

$$\sum k_i = k,$$

there is a realization of G containing edge-disjoint subgraphs $F_1, \ldots, F_t$ such that each F_i is a k_i-factor of G.

Odd Order

If n is odd, then each k_i (and hence k) must be even.

Since any $2r$-regular graph has a 2-factorization, the conjecture for n odd follows from Kundu's Theorem.

Odd Order

If n is odd, then each k_i (and hence k) must be even.

Since any $2r$-regular graph has a 2-factorization, the conjecture for n odd follows from Kundu's Theorem.

It would therefore be sufficient to prove the following:

Conj. Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$.

Then there exists a realization G of π that contains
k edge-disjoint 1-factors.

Bounded Maximum Degree

It is straightforward to verify the conjecture when the largest term in π is bounded.

Thm. Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k > 0$

$$\text{and } \max d_i \leq \frac{n}{2} + 1.$$

Then there exists a realization G of π that contains k edge-disjoint 1-factors.

Proof

The proof is by induction on k, with $k = 1$ following from Kundu's Theorem.

Proof

The proof is by induction on k, with $k = 1$ following from Kundu's Theorem.

Suppose that $\pi = (d_1, ..., d_n)$ and $\pi - k = (d_1 - k, ..., d_n - k)$ are both graphic, $k \geq 2$.

By Rao and Rao, $\pi - 2 = (d_1 - 2, ..., d_n - 2)$ is graphic.

By induction, there is a realization $G = G(\pi - 2)$ with $k - 2$ edge-disjoint 1-factors.

Proof

The proof is by induction on k, with $k = 1$ following from Kundu's Theorem.

Suppose that $\pi = (d_1, \ldots, d_n)$ and $\pi - k = (d_1 - k, \ldots, d_n - k)$ are both graphic, $k \geq 2$.

By Rao and Rao, $\pi - 2 = (d_1 - 2, \ldots, d_n - 2)$ is graphic.

By induction, there is a realization $G = G(\pi - 2)$ with $k - 2$ edge-disjoint 1-factors.

$$\Delta(G) \leq \frac{n}{2} - 1 \Longrightarrow \delta(\overline{G}) \geq \frac{n}{2} \Longrightarrow \overline{G} \text{ is hamiltonian.}$$

Decompose hamiltonian cycle in $\overline{G}$ into two 1-factors; add these to G.

k=2

Thm. Let n be even and let $\pi = (d_1, ..., d_n)$ be a graphic seq. If $\pi - 2 = (d_1 - 2, ..., d_n - 2)$ is graphic, then there is a realization of π that contains two edge-disjoint 1-factors.

k=2

Thm. Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq. If $\pi - 2 = (d_1 - 2, \ldots, d_n - 2)$ is graphic, then there is a realization of π that contains two edge-disjoint 1-factors.

Proof.

By Kundu, there is a G of π with a 2-factor F.

If every component of F has even order, then F decomposes into two edge-disjoint 1-factors, and the result follows.

$k=2$

Thm. Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq. If $\pi - 2 = (d_1 - 2, \ldots, d_n - 2)$ is graphic, then there is a realization of π that contains two edge-disjoint 1-factors.

Proof.

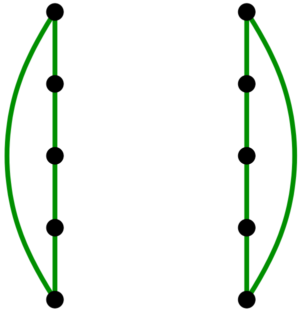
By Kundu, there is a G of π with a 2-factor F.

If every component of F has even order, then F decomposes into two edge-disjoint 1-factors, and the result follows.

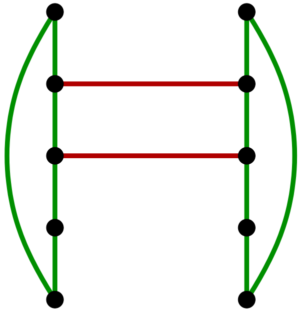
Suppose not. Since n is even, F must have at least two odd components, C_{2i+1} and C_{2j+1}.

All of the edges between C_{2i+1} and C_{2j+1} must be in G or $H = \overline{G}$.

$k=2$

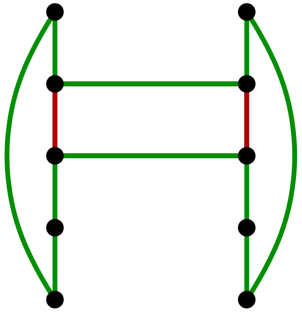


k=2



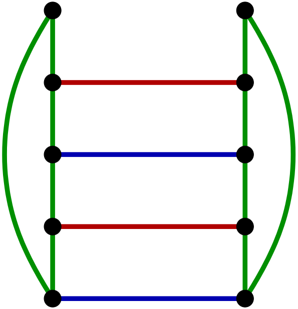
We can do a 2-switch to reduce the number of odd cyles in F.

k=2



We can do a 2-switch to reduce the number of odd cyles in F.

k=2



The edges between C_{2i+1} and C_{2j+1} cannot alternate between G and the complement of G.

(k-1)-factors and 1-factors

Thm. Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k \geq 1$.

Then there exists a realization of π containing a 1-factor F_1 and a $(k-1)$-factor F_{k-1} that are edge-disjoint.

(k-1)-factors and 1-factors

Thm. Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k \geq 1$.

Then there exists a realization of π containing a 1-factor F_1 and a $(k-1)$-factor F_{k-1} that are edge-disjoint.

Lem. Let F be a k-regular graph with an even number of vertices, and let (A, B, C) be the Gallai-Edmonds Decomposition of F.

If F does **not** contain a 1-factor, then there exists a maximum matching M of F that misses two nontrivial factor-critical components of $F[A]$.

(k-1)-factors and 1-factors

Thm. Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k \geq 1$.

Then there exists a realization of π containing a 1-factor F_1 and a $(k-1)$-factor F_{k-1} that are edge-disjoint.

Lem. Let F be a k-regular graph with an even number of vertices, and let (A, B, C) be the Gallai-Edmonds Decomposition of F.

If F does **not** contain a 1-factor, then there exists a maximum matching M of F that misses two nontrivial factor-critical components of $F[A]$.

These components must contain odd cycles,
and we apply the previous technique.

k=3

Thm. Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq. If $\pi - 3 = (d_1 - 3, \ldots, d_n - 3)$ is graphic, then there is a realization of π that contains three edge-disjoint 1-factors.

k=3

Thm. Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq. If $\pi - 3 = (d_1 - 3, \ldots, d_n - 3)$ is graphic, then there is a realization of π that contains three edge-disjoint 1-factors.

Proof.

By the previous theorem, there is a realization G of π with an edge-disjoint matching M and 2-factor F.

If every component of F has even order, then F decomposes into two edge-disjoint 1-factors, and the result follows.

k=3

Thm. Let n be even and let $\pi = (d_1, ..., d_n)$ be a graphic seq. If $\pi - 3 = (d_1 - 3, ..., d_n - 3)$ is graphic, then there is a realization of π that contains three edge-disjoint 1-factors.

Proof.

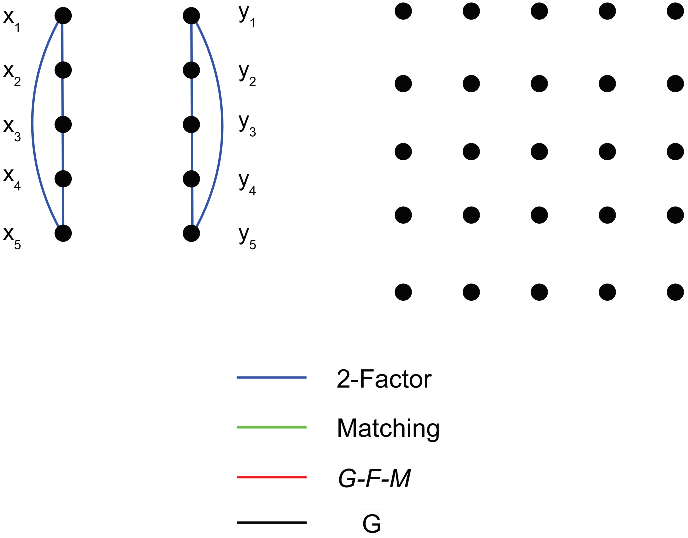
By the previous theorem, there is a realization G of π with an edge-disjoint matching M and 2-factor F.

If every component of F has even order, then F decomposes into two edge-disjoint 1-factors, and the result follows.

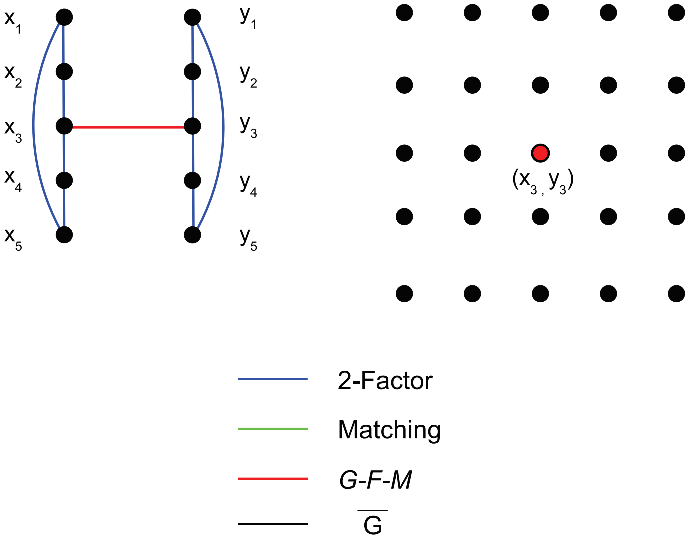
Suppose not. Since n is even, F must have at least two odd components, C_{2i+1} and C_{2j+1}.

We create an auxiliary graph Aux on a $(2i+1) \times (2j+1)$ grid with each vertex representing an edge between the odd components.

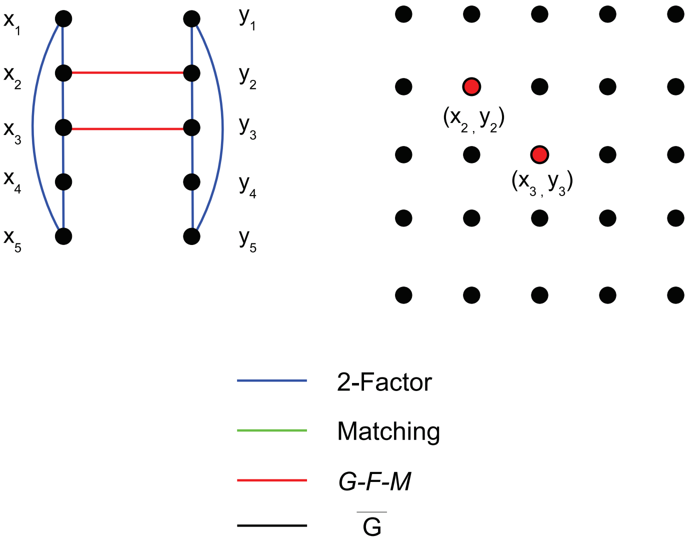
Proof



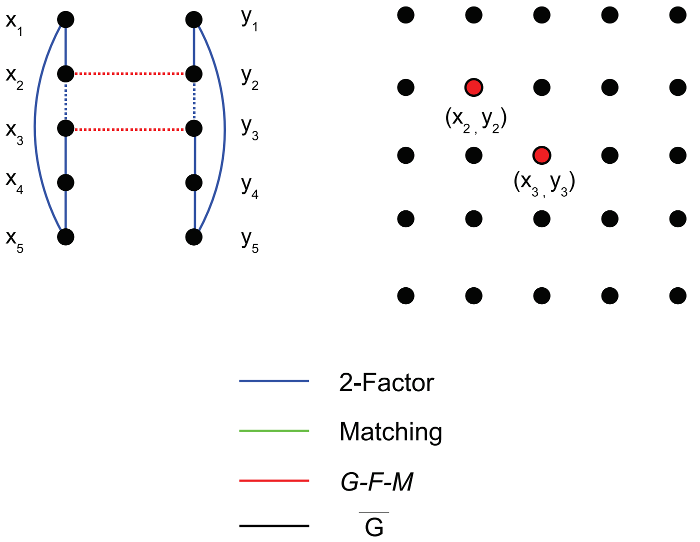
Proof



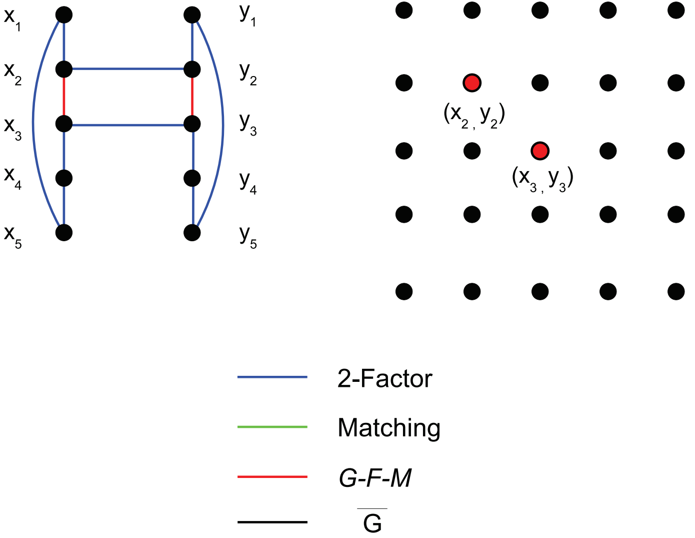
Proof



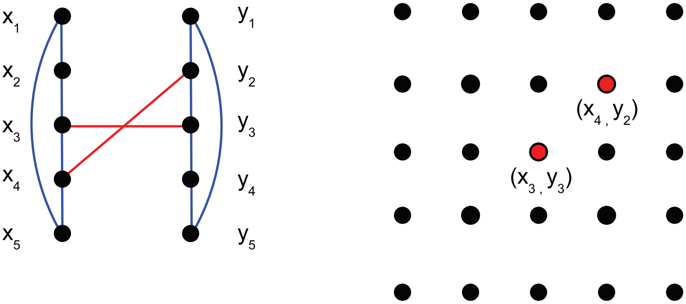
Proof



Proof



Proof



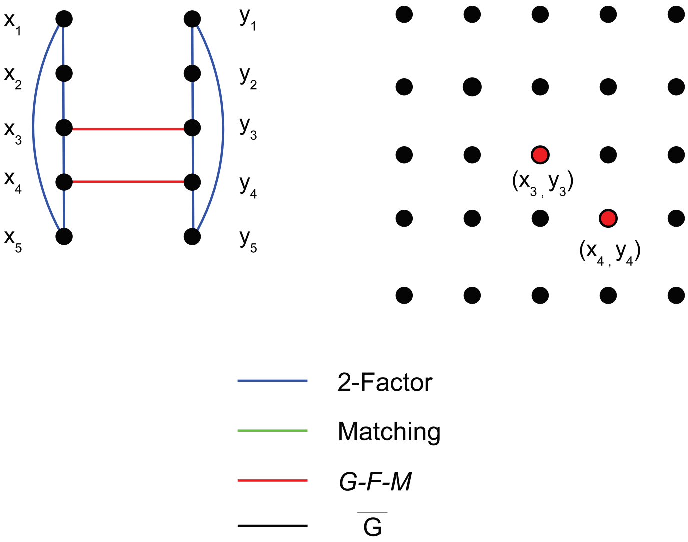
2-Factor

Matching

G-F-M

$\overline{G}$

Proof



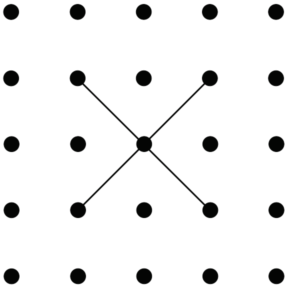
Proof

If any of the edges x_2y_2, x_4y_2, x_2y_4, or x_4y_4 were the same color (ie, in the same subgraph) as x_3y_3, then we could reduce the number of odd components in F.

Proof

If any of the edges x_2y_2, x_4y_2, x_2y_4, or x_4y_4 were the same color (ie, in the same subgraph) as x_3y_3, then we could reduce the number of odd components in F.

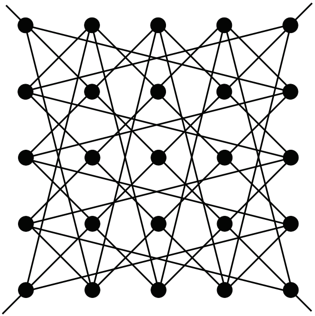
We add edges to the auxiliary graph Aux based on these "forbidden" edge swaps.



Proof

If any of the edges x_2y_2, x_4y_2, x_2y_4, or x_4y_4 were the same color (ie, in the same subgraph) as x_3y_3, then we could reduce the number of odd components in F.

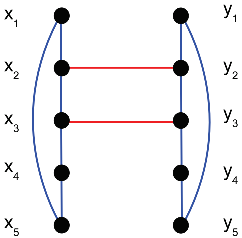
We add edges to the auxiliary graph Aux based on these "forbidden" edge swaps.

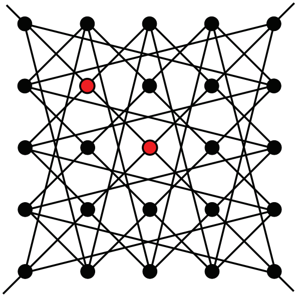


Proof

Can we properly color *Aux* with green (M), red ($G - F - M$), and black ($\overline{G}$)?

If we cannot, then an edge swap exists to help “fix” F.

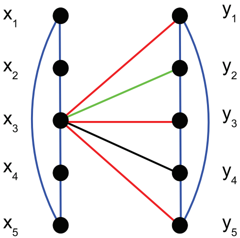


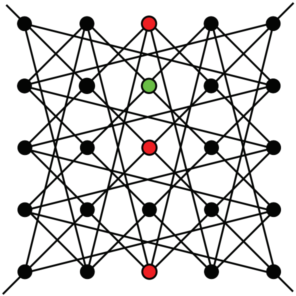


Proof

Each row and column in Aux corresponds to a vertex in either C_{2i+1} or C_{2j+1}.

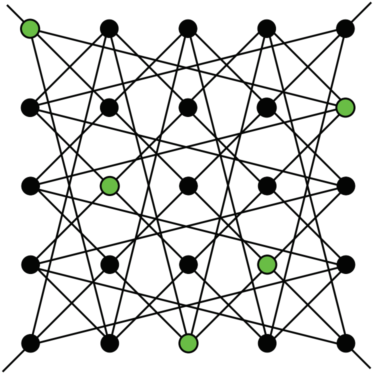
Hence green for the matching M can be used at most once in any row or column.





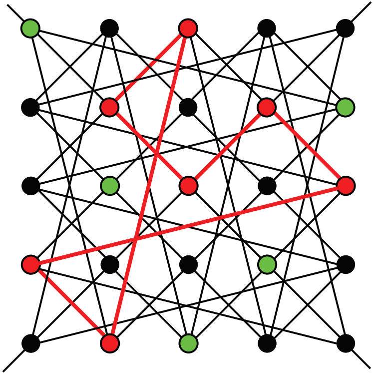
Proof

We conclude the proof by showing that if at most one vertex in each row and column of *Aux* is colored green, and no two of these green vertices are adjacent, then there is an odd cycle amongst the other vertices.



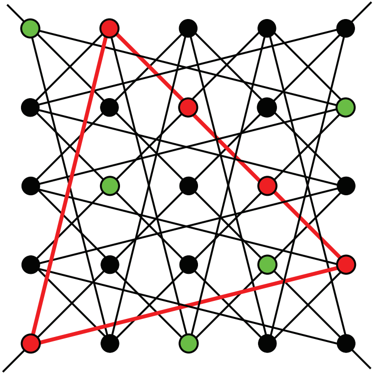
Proof

We conclude the proof by showing that if at most one vertex in each row and column of *Aux* is colored green, and no two of these green vertices are adjacent, then there is an odd cycle amongst the other vertices.



Proof

We conclude the proof by showing that if at most one vertex in each row and column of *Aux* is colored green, and no two of these green vertices are adjacent, then there is an odd cycle amongst the other vertices.



(k-2)-factors and two 1-factors

Thm. Let n be even and let $\pi = (d_1, \ldots, d_n)$ be a graphic seq such that $\pi - k = (d_1 - k, \ldots, d_n - k)$ is graphic for some $k \geq 1$.

Then there exists a realization of π containing two 1-factors F_1 and F_2 and a $(k-2)$-factor F_{k-2} that are edge-disjoint.

The proof follows from the Gallai-Edmonds lemma and the previous theorem.

Future Work

- Characterize when degree sequences pack.
- Find sufficient conditions for when degree seqs pack.
- Prove the conjecture generalizing Kundu's Theorem.

  Our techniques will not go beyond $k = 4$.

Packing Degree Sequences

Stephen G. Hartke

Department of Mathematics

University of Nebraska–Lincoln

www.math.unl.edu/∼shartke2

hartke@math.unl.edu

Joint work with Art Busch, Mike Ferrara, Mike Jacobson,

Hemanshu Kaul, Tyler Seacrest, and Doug West