Independence polynomials of path-like graphs

Patrick Bahls
University of North Carolina, Asheville
facstaff.unca.edu/pbahls

Joint work with
Nathan Salazar
Western Kentucky University

Outline of our discussion

1. Definitions
2. Statement of theorems
3. Proof sketches
4. Directions for future study

Outline of our discussion

1. Definitions
2. Statement of theorems
3. Proof sketches
4. Directions for future study

Outline of our discussion

1. Definitions
2. Statement of theorems
3. Proof sketches
4. Directions for future study

Outline of our discussion

1. Definitions
2. Statement of theorems
3. Proof sketches
4. Directions for future study

Definitions 1: independence

A set of vertices in G is called *independent* if no two of its vertices are adjacent.

Definitions 1: independence

A set of vertices in G is called *independent* if no two of its vertices are adjacent.

The *independence number*, $\alpha(G)$, of a graph G is the cardinality of a maximal independent set of vertices in G.

Definitions 1: independence

A set of vertices in G is called *independent* if no two of its vertices are adjacent.

The *independence number*, $\alpha(G)$, of a graph G is the cardinality of a maximal independent set of vertices in G.

The *independence polynomial*, $I(G;x)$, of a graph G is defined by

$$I(G;x) = \sum_{i=0}^{\alpha(G)} a_i x^i,$$

where a_i is the number of independent sets of vertices of cardinality i.

Definitions 2: properties of polynomials

Let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial.

Definitions 2: properties of polynomials

Let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial.

1. p is called *unimodal* if there exists a value k such that

$$a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n.$$

Definitions 2: properties of polynomials

Let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial.

1. p is called *unimodal* if there exists a value k such that
$$a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n.$$
2. p is called *symmetric* if $a_i = a_{n-i}$ for all $i = 0, 1, ..., n$.

Definitions 2: properties of polynomials

Let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial.

1. p is called *unimodal* if there exists a value k such that
$$a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n.$$
2. p is called *symmetric* if $a_i = a_{n-i}$ for all $i = 0, 1, ..., n$.
3. p is called *logarithmically concave* (or simply *log-concave*) if
$$a_i^2 \geq a_{i-1}a_{i+1} \text{ for all } i = 1, 2, ..., n-1.$$

Definitions 3: relationships between these properties

If p is both symmetric and unimodal, we will say that it is *SU*. One of the simplest yet most important SU polynomials is $(x+1)^n$.

Definitions 3: relationships between these properties

If p is both symmetric and unimodal, we will say that it is *SU*. One of the simplest yet most important SU polynomials is $(x + 1)^n$.

Proposition. If p is log-concave, then it is unimodal.

Definitions 3: relationships between these properties

If p is both symmetric and unimodal, we will say that it is *SU*. One of the simplest yet most important SU polynomials is $(x+1)^n$.

Proposition. If p is log-concave, then it is unimodal.

The following fact will be useful:

Theorem 0. If all of the roots of $p = \sum a_i x^i$ are real, then the sequence $(a_i/\binom{n}{i})$ is log-concave. (As a consequence, p itself is log-concave.)

Definitions 4: graphs of interest

For $k \geq 1$ and $t \geq 2$, $P(t,k)$ will denote the K_t-*path* of length k, the graph $G = (V, E)$ with

Definitions 4: graphs of interest

For $k \geq 1$ and $t \geq 2$, $P(t,k)$ will denote the K_t-*path* of length k, the graph $G = (V, E)$ with

1. $V = \{v_1, v_2, ..., v_{t+k-1}\}$ and

Definitions 4: graphs of interest

For $k \geq 1$ and $t \geq 2$, $P(t,k)$ will denote the K_t-*path* of length k, the graph $G = (V, E)$ with

1. $V = \{v_1, v_2, ..., v_{t+k-1}\}$ and
2. $E = \{\{v_i, v_{i+j}\} \mid 1 \leq i \leq t+k-2, 1 \leq j \leq \min\{t-1, t+k-i-1\}\}$.

Definitions 4: graphs of interest

For $k \geq 1$ and $t \geq 2$, $P(t,k)$ will denote the K_t-*path* of length k, the graph $G = (V, E)$ with

1. $V = \{v_1, v_2, ..., v_{t+k-1}\}$ and
2. $E = \{\{v_i, v_{i+j}\} \mid 1 \leq i \leq t+k-2, 1 \leq j \leq \min\{t-1, t+k-i-1\}\}$.

$P(t,k)$ is formed by gluing k copies of K_t along subgraphs isomorphic to K_{t-1}.

Definitions 5: more graphs of interest

Now let $d \geq 0$. We create the *d-augmented K_t-path* $P(t, k, d)$ by introducing to $P(t, k)$

Definitions 5: more graphs of interest

Now let $d \geq 0$. We create the *d-augmented K_t-path* $P(t,k,d)$ by introducing to $P(t,k)$

1. new vertices,

$$\bigcup_{i=0}^{t+k-2} \{u_{i,1}, u_{i,2}, ..., u_{i,d}\}, \text{ and}$$

Definitions 5: more graphs of interest

Now let $d \geq 0$. We create the *d-augmented K_t-path* $P(t,k,d)$ by introducing to $P(t,k)$

1. new vertices,

$$\bigcup_{i=0}^{t+k-2} \{u_{i,1}, u_{i,2}, ..., u_{i,d}\}, \text{ and}$$

2. new edges,

$$\bigcup_{i=1}^{t+k-2} \{\{v_i, u_{i,j}\}, \{v_{i+1}, u_{i,j}\} \mid j = 1, ..., d\} \bigcup \{\{v_1, u_{0,j}\} \mid j = 1, ...d\}.$$

Definitions 6: still more graphs of interest

Let $G = (V, E)$ be given, and let $U \subseteq V$. For $v \notin V$, the graph $G^*(U, v)$ called the *cone of G on U*, is defined by

Definitions 6: still more graphs of interest

Let $G = (V, E)$ be given, and let $U \subseteq V$. For $v \notin V$, the graph $G^*(U, v)$ called the *cone of G on U*, is defined by

1. $V(G^*(U, V)) = V \cup \{v\}$, and

Definitions 6: still more graphs of interest

Let $G = (V, E)$ be given, and let $U \subseteq V$. For $v \notin V$, the graph $G^*(U, v)$ called the *cone of G on U*, is defined by

1. $V(G^*(U, V)) = V \cup \{v\}$, and
2. $E(G^*(U, v)) = E \cup \{\{u, v\} \mid u \in U\}$.

Definitions 6: still more graphs of interest

Let $G = (V, E)$ be given, and let $U \subseteq V$. For $v \notin V$, the graph $G^*(U, v)$ called the *cone of G on U*, is defined by

1. $V(G^*(U, V)) = V \cup \{v\}$, and
2. $E(G^*(U, v)) = E \cup \{\{u, v\} \mid u \in U\}$.

For a graph Γ, we denote by $\Gamma\nabla(G, U)$ the graph formed by coning G on U at each of Γ's vertices.

Known results

1. In [Zhu 2007], Z.-F. Zhu establishes symmetry and unimodality of

$$I(P_k\nabla(N_2, V(N_2)); x),$$

   corresponding to 2-regular *caterpillars*. (N_s is the null graph on s vertices.)

Known results

1. In [Zhu 2007], Z.-F. Zhu establishes symmetry and unimodality of
   $$I(P_k\nabla(N_2, V(N_2)); x),$$
   corresponding to 2-regular *caterpillars*. (N_s is the null graph on s vertices.)

2. In various papers from 2004 through 2008, V.E. Levit and E. Mandrescu establish symmetry and unimodality for a number of graphs not closely related to ours.

Known results

1. In [Zhu 2007], Z.-F. Zhu establishes symmetry and unimodality of
$$I(P_k \nabla (N_2, V(N_2)); x),$$
corresponding to 2-regular *caterpillars*. (N_s is the null graph on s vertices.)

2. In various papers from 2004 through 2008, V.E. Levit and E. Mandrescu establish symmetry and unimodality for a number of graphs not closely related to ours.

3. The closest they come to our results is in [Levit-Mandrescu 2007], in which a very *specific* “path-like” graph family is considered.

Theorem 1: Symmetry and Unimodality

Theorem 1. Let $d \geq 0$, $k \geq 1$, and $t \geq 2$. Let $G = (V, E)$ and suppose $U \subseteq V$ such that each of the polynomials

Theorem 1: Symmetry and Unimodality

Theorem 1. Let $d \geq 0$, $k \geq 1$, and $t \geq 2$. Let $G = (V, E)$ and suppose $U \subseteq V$ such that each of the polynomials

- $b(x) = I(G; x)$,
- $f(x) = I(G^*(U, v); x)$, and
- $c(x) = I(G - U; x)$

Theorem 1: Symmetry and Unimodality

Theorem 1. Let $d \geq 0$, $k \geq 1$, and $t \geq 2$. Let $G = (V, E)$ and suppose $U \subseteq V$ such that each of the polynomials

- $b(x) = I(G; x)$,
- $f(x) = I(G^*(U, v); x)$, and
- $c(x) = I(G - U; x)$

is SU. Finally, suppose $\deg(b) = \deg(f) = \deg(c) + 2$. Then the graph $P(t, k, d)\nabla(G, U)$ is SU as well.

Theorem 2: Log-concavity

Theorem 2. Let $k \geq 1$ and $t \geq 2$. Let $G = (V, E)$ and suppose $U \subseteq V$ such that

Theorem 2: Log-concavity

Theorem 2. Let $k \geq 1$ and $t \geq 2$. Let $G = (V, E)$ and suppose $U \subseteq V$ such that

1. $\deg(b) = \deg(c) + 2$, where as before $b(x) = I(G; x)$ and $c(x) = I(G - U; x)$,

Theorem 2: Log-concavity

Theorem 2. Let $k \geq 1$ and $t \geq 2$. Let $G = (V, E)$ and suppose $U \subseteq V$ such that

1. $\deg(b) = \deg(c) + 2$, where as before $b(x) = I(G; x)$ and $c(x) = I(G - U; x)$,
2. $c|b$ in $\mathbb{Z}[x]$, and

Theorem 2: Log-concavity

Theorem 2. Let $k \geq 1$ and $t \geq 2$. Let $G = (V, E)$ and suppose $U \subseteq V$ such that

1. $\deg(b) = \deg(c) + 2$, where as before $b(x) = I(G; x)$ and $c(x) = I(G - U; x)$,
2. $c|b$ in $\mathbb{Z}[x]$, and
3. every root of b (and thus also of c) is real.

Theorem 2: Log-concavity

Theorem 2. Let $k \geq 1$ and $t \geq 2$. Let $G = (V, E)$ and suppose $U \subseteq V$ such that

1. $\deg(b) = \deg(c) + 2$, where as before $b(x) = I(G; x)$ and $c(x) = I(G - U; x)$,
2. $c|b$ in $\mathbb{Z}[x]$, and
3. every root of b (and thus also of c) is real.

Then $P(t, k)\nabla(G, U)$ is log-concave.

Useful formulas

The following facts will prove useful in computing $I(G; x)$ for our augmented paths.

Useful formulas

The following facts will prove useful in computing $I(G; x)$ for our augmented paths.

Proposition. Let G_1 and G_2 be disjoint graphs. Then

$$I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x).$$

Useful formulas

The following facts will prove useful in computing $I(G;x)$ for our augmented paths.

Proposition. Let G_1 and G_2 be disjoint graphs. Then

$$I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x).$$

Proposition. Let $G = (V, E)$ and suppose $v \in V$. Then

$$I(G; x) = I(G - v; x) + x \cdot I(G - N[v]; x),$$

where $N[v]$ is the *closed neighborhood* of the vertex v in G.

Toward a proof of Theorem 1

The following lemma will be helpful in proving Theorem 1:

Toward a proof of Theorem 1

The following lemma will be helpful in proving Theorem 1:

Lemma. Let p and q be polynomials of degree r and $r-1$, respectively, for some $r \geq 2$. Suppose that $p(0) \neq 0$ and $q(0) = 0$. If p and q are SU, then so is $p + q$.

Toward a proof of Theorem 1

The following lemma will be helpful in proving Theorem 1:

Lemma. Let p and q be polynomials of degree r and $r-1$, respectively, for some $r \geq 2$. Suppose that $p(0) \neq 0$ and $q(0) = 0$. If p and q are SU, then so is $p + q$.

Proof. This fact is trivial: one merely notes that the peaks of p and q "match up" when the sum $p + q$ is computed.

Toward a proof of Theorem 1

The following lemma will be helpful in proving Theorem 1:

Lemma. Let p and q be polynomials of degree r and $r-1$, respectively, for some $r \geq 2$. Suppose that $p(0) \neq 0$ and $q(0) = 0$. If p and q are SU, then so is $p + q$.

Proof. This fact is trivial: one merely notes that the peaks of p and q "match up" when the sum $p + q$ is computed.

Note. This lemma will be applied repeatedly to the two terms on the right-hand side of the recurrence relation in the second proposition.

The structure of the subgraph

Lemma. Let t, k, d, G, and U be as in the hypotheses of Theorem 1, and let v_1 be the first vertex in the K_t path underlying $P(t, k, d)$.

The structure of the subgraph

Lemma. Let t, k, d, G, and U be as in the hypotheses of Theorem 1, and let v_1 be the first vertex in the K_t path underlying $P(t,k,d)$.

1. If $k \geq 2$, then

$$(P(t,k,d)\nabla(G,U)) - v_1 \cong G \cup dG^*(U) \cup (P(t,k-1,d)\nabla(G,U)).$$

The structure of the subgraph

Lemma. Let t, k, d, G, and U be as in the hypotheses of Theorem 1, and let v_1 be the first vertex in the K_t path underlying $P(t,k,d)$.

1. If $k \geq 2$, then

$$(P(t,k,d)\nabla(G,U)) - v_1 \cong G \cup dG^*(U) \cup (P(t,k-1,d)\nabla(G,U)).$$

2. If $k \geq t+1$, then $(P(t,k,d)\nabla(G,U)) - N[v_1]$ is isomorphic to

$$(G-U) \cup (2d+t-1)G \cup d(t-2)G^*(U) \cup (P(t,k-t,d)\nabla(G,U)).$$

The structure of the subgraph

Lemma. Let t, k, d, G, and U be as in the hypotheses of Theorem 1, and let v_1 be the first vertex in the K_t path underlying $P(t,k,d)$.

1. If $k \geq 2$, then

   $$(P(t,k,d)\nabla(G,U)) - v_1 \cong G \cup dG^*(U) \cup (P(t,k-1,d)\nabla(G,U)).$$

2. If $k \geq t+1$, then $(P(t,k,d)\nabla(G,U)) - N[v_1]$ is isomorphic to

   $$(G-U) \cup (2d+t-1)G \cup d(t-2)G^*(U) \cup (P(t,k-t,d)\nabla(G,U)).$$

(For a graph Γ and nonnegative integer r, $r\Gamma$ denotes the union of r disjoint copies of Γ.)

Applying these facts...

The preceding lemma allows us to apply our recurrence relations governing the independence polynomials I.

Applying these facts...

The preceding lemma allows us to apply our recurrence relations governing the independence polynomials I.

For fixed t and d, let $p_k(x) = I(P(t, k, d)\nabla(G, U); x)$. If k is large enough, then the previous lemma tells us

Applying these facts...

The preceding lemma allows us to apply our recurrence relations governing the independence polynomials I.

For fixed t and d, let $p_k(x) = I(P(t,k,d)\nabla(G,U);x)$. If k is large enough, then the previous lemma tells us

$$p_k(x) = bf^d p_{k-1}(x) + xcb^{2d+t-1} f^{d(t-2)} p_{k-t}(x),$$

for $b(x) = I(G;x)$, $c(x) = I(G-U;x)$, and $f(x) = I(G^*(U,v);x)$, as before.

Inductive step

Inductively assume that $\deg(p_k) = (t + k - 1)(d + 1)\deg(b)$. Then

Inductive step

Inductively assume that $\deg(p_k) = (t+k-1)(d+1)\deg(b)$. Then

$$\deg(bf^d p_{k-1}) = (d+1)\deg(b) + (t+k-2)(d+1)\deg(b)$$

Inductive step

Inductively assume that $\deg(p_k) = (t + k - 1)(d + 1)\deg(b)$. Then

$$\begin{aligned} \deg(bf^d p_{k-1}) &= (d+1)\deg(b) + (t+k-2)(d+1)\deg(b) \\ &= (t+k-1)(d+1)\deg(b) \end{aligned}$$

Inductive step

Inductively assume that $\deg(p_k) = (t+k-1)(d+1)\deg(b)$. Then

$$\begin{aligned} \deg(bf^d p_{k-1}) &= (d+1)\deg(b) + (t+k-2)(d+1)\deg(b) \\ &= (t+k-1)(d+1)\deg(b) \end{aligned}$$

and

$$\deg(xcb^{2d+t-1} f^{d(t-2)} p_{k-t})$$

Inductive step

Inductively assume that $\deg(p_k) = (t+k-1)(d+1)\deg(b)$. Then

$$\begin{aligned}
\deg(bf^d p_{k-1}) &= (d+1)\deg(b) + (t+k-2)(d+1)\deg(b) \\
&= (t+k-1)(d+1)\deg(b)
\end{aligned}$$

and

$$\begin{aligned}
&\deg(xcb^{2d+t-1}f^{d(t-2)}p_{k-t}) \\
= \; &\deg(c) + (td+t-1)\deg(b) + (k-1)(d+1)\deg(b) + 1
\end{aligned}$$

Inductive step

Inductively assume that $\deg(p_k) = (t+k-1)(d+1)\deg(b)$. Then

$$\begin{aligned}\deg(bf^d p_{k-1}) &= (d+1)\deg(b) + (t+k-2)(d+1)\deg(b)\\ &= (t+k-1)(d+1)\deg(b)\end{aligned}$$

and

$$\begin{aligned}&\deg(xcb^{2d+t-1}f^{d(t-2)}p_{k-t})\\ =\ &\deg(c) + (td+t-1)\deg(b) + (k-1)(d+1)\deg(b) + 1\\ =\ &(t+k-1)(d+1)\deg(b) - 1.\end{aligned}$$

Done!

An application of our lemma concerning the sums of SU polynomials which differ in degree by 1 finishes the proof.

Done!

An application of our lemma concerning the sums of SU polynomials which differ in degree by 1 finishes the proof.

Note. The additional inductive assumption on degrees is satisfied as well.

Examples

Example 1. $G = K_s - e, U = V(G)$. Then $b(x) = I(G; x) = x^2 + sx + 1$ and $c(x) = I(G - U; x) = 1$ are both SU, so $P(t, k, d)\nabla(G, U)$ is SU in this case.

Examples

Example 1. $G = K_s - e, U = V(G)$. Then $b(x) = I(G; x) = x^2 + sx + 1$ and $c(x) = I(G - U; x) = 1$ are both SU, so $P(t, k, d)\nabla(G, U)$ is SU in this case.

Example 2. If $G = P(2, k, 0)\nabla(N_2, V(N_2))$ is *itself* an augmented 2-path, and U consists of one of the path's peripheral vertices and its pendant vertices, then $G - U = P(2, k - 1, 0)\nabla(N_2, V(N_2))$ gives rise to an independence polynomial of degree $\deg(I(G; x)) - 2$, and the hypotheses of Theorem 1 are satisfied.

Examples

Example 1. $G = K_s - e, U = V(G)$. Then $b(x) = I(G; x) = x^2 + sx + 1$ and $c(x) = I(G - U; x) = 1$ are both SU, so $P(t, k, d)\nabla(G, U)$ is SU in this case.

Example 2. If $G = P(2, k, 0)\nabla(N_2, V(N_2))$ is *itself* an augmented 2-path, and U consists of one of the path's peripheral vertices and its pendant vertices, then $G - U = P(2, k - 1, 0)\nabla(N_2, V(N_2))$ gives rise to an independence polynomial of degree $\deg(I(G; x)) - 2$, and the hypotheses of Theorem 1 are satisfied.

Examples $\geq$ 3. Finding examples in which $\deg(b) = \deg(f) = 2s$, $s \geq 1$, is easy; finding examples in which $\deg(b) = \deg(f) = 2s + 1$, $s \geq 2$, is harder.

A related construction

The same technique can be used to show

Proposition. Let $k \geq 1$ be an integer, and let b, f, and c be polynomials as in Theorem 1. Then the graph $C_k \nabla (G, U)$ is SU, where C_k is the *ladder* of length k.

A related construction

The same technique can be used to show

Proposition. Let $k \geq 1$ be an integer, and let b, f, and c be polynomials as in Theorem 1. Then the graph $C_k \nabla (G, U)$ is SU, where C_k is the *ladder* of length k.

The proof is nearly identical to the above, with the appropriate choice of removed vertex v in the inductive step.

Toward a proof of Theorem 2

The following fact will be very useful:

Proposition S. Let p and q be polynomials with positive integer coefficients such that

Toward a proof of Theorem 2

The following fact will be very useful:

Proposition S. Let p and q be polynomials with positive integer coefficients such that (1) $\deg(p) = k$ and $\deg(q) = k + 2$ for some k even; (2) the roots α_i of p and β_i of q are all real and distinct; and

Toward a proof of Theorem 2

The following fact will be very useful:

Proposition S. Let p and q be polynomials with positive integer coefficients such that (1) $\deg(p) = k$ and $\deg(q) = k + 2$ for some k even; (2) the roots α_i of p and β_i of q are all real and distinct; and (3)

$$\begin{array}{cccccccccccccc} \beta_1 & < & \alpha_1 & < & \cdots & < & \beta_{\frac{k}{2}} & < & \alpha_{\frac{k}{2}} & < & \beta_{\frac{k+2}{2}} & & & \\ & < & \beta_{\frac{k+4}{2}} & < & \alpha_{\frac{k+2}{2}} & < & \beta_{\frac{k+6}{2}} & < & \cdots & < & \alpha_k & < & \beta_{k+2} & < \; 0. \end{array}$$

Toward a proof of Theorem 2

The following fact will be very useful:

Proposition S. Let p and q be polynomials with positive integer coefficients such that (1) $\deg(p) = k$ and $\deg(q) = k + 2$ for some k even; (2) the roots α_i of p and β_i of q are all real and distinct; and (3)

$$\begin{aligned} \beta_1 &< \alpha_1 < \cdots < \beta_{\frac{k}{2}} < \alpha_{\frac{k}{2}} < \beta_{\frac{k+2}{2}} \\ &< \beta_{\frac{k+4}{2}} < \alpha_{\frac{k+2}{2}} < \beta_{\frac{k+6}{2}} < \cdots < \alpha_k < \beta_{k+2} < 0. \end{aligned}$$

Then the polynomial $q + xp$ has $k + 2$ distinct real roots.

Proof

Under the hypotheses of Proposition S the roots α_i, β_i, and 0 determine $2k+3$ intervals, on n of which q and xp have opposite signs.

Proof

Under the hypotheses of Proposition S the roots α_i, β_i, and 0 determine $2k+3$ intervals, on n of which q and xp have opposite signs.

On such intervals (and only on these) the sum $q + xp$ will have a root.

Proof

Under the hypotheses of Proposition S the roots α_i, β_i, and 0 determine $2k + 3$ intervals, on n of which q and xp have opposite signs.

On such intervals (and only on these) the sum $q + xp$ will have a root.

Distinctness of the roots follows as well.

Proof

Under the hypotheses of Proposition S the roots α_i, β_i, and 0 determine $2k + 3$ intervals, on n of which q and xp have opposite signs.

On such intervals (and only on these) the sum $q + xp$ will have a root.

Distinctness of the roots follows as well.

Note. A similar result holds for k odd.

Proof of Theorem 2 ($t = 2$)

We will describe a proof of the case $t = 2$, in which the path $P(t, k)$ is an ordinary path.

Proof of Theorem 2 ($t = 2$)

We will describe a proof of the case $t = 2$, in which the path $P(t, k)$ is an ordinary path.

Fix G and U (and thus b and c) as in the statement of Theorem 2. Let $p_k = p_{2,k,G,U}$ be the independence polynomial of $P(2, k)\nabla(G, U)$.

Proof of Theorem 2 ($t = 2$)

We will describe a proof of the case $t = 2$, in which the path $P(t, k)$ is an ordinary path.

Fix G and U (and thus b and c) as in the statement of Theorem 2. Let $p_k = p_{2,k,G,U}$ be the independence polynomial of $P(2, k)\nabla(G, U)$.

We must show that every root of p_k is real, for all $k \geq 1$.

A formula for p_k

Using one of our recurrences for independence polynomials, we obtain

Proposition. Let p_k be as above. Then

A formula for p_k

Using one of our recurrences for independence polynomials, we obtain

Proposition. Let p_k be as above. Then

1. $p_1 = b + cx$,

A formula for p_k

Using one of our recurrences for independence polynomials, we obtain

Proposition. Let p_k be as above. Then

1. $p_1 = b + cx$,
2. $p_2 = bp_1 + xcb$, and

A formula for p_k

Using one of our recurrences for independence polynomials, we obtain

Proposition. Let p_k be as above. Then

1. $p_1 = b + cx$,
2. $p_2 = bp_1 + xcb$, and
3. $p_k = b(p_{k-1} + xcp_{k-2})$ for $k \geq 3$.

A formula for p_k

Using one of our recurrences for independence polynomials, we obtain

Proposition. Let p_k be as above. Then

1. $p_1 = b + cx$,
2. $p_2 = bp_1 + xcb$, and
3. $p_k = b(p_{k-1} + xcp_{k-2})$ for $k \geq 3$.

Thus $\deg(p_k) = n \deg(b)$ for all $k \geq 1$.

A simpler sequence of polynomials

Since every root of b (and c) is real, we will simplify p_k by dividing out as many powers of b as we can.

A simpler sequence of polynomials

Since every root of b (and c) is real, we will simplify p_k by dividing out as many powers of b as we can.

For polynomials $f(x)$ and $g(x)$ we write $f^n \| g$ to mean that f^n divides g in $\mathbb{Z}[x]$, but that no higher power of f divides g.

A simpler sequence of polynomials

Since every root of b (and c) is real, we will simplify p_k by dividing out as many powers of b as we can.

For polynomials $f(x)$ and $g(x)$ we write $f^n \| g$ to mean that f^n divides g in $\mathbb{Z}[x]$, but that no higher power of f divides g.

Lemma. Let $k \geq 1$. Then $b^{\lfloor k/2 \rfloor} \| p_k$.

A simpler sequence of polynomials

Since every root of b (and c) is real, we will simplify p_k by dividing out as many powers of b as we can.

For polynomials $f(x)$ and $g(x)$ we write $f^n \| g$ to mean that f^n divides g in $\mathbb{Z}[x]$, but that no higher power of f divides g.

Lemma. Let $k \geq 1$. Then $b^{\lfloor k/2 \rfloor} \| p_k$.

Proof. Induction.

A simpler sequence of polynomials

Since every root of b (and c) is real, we will simplify p_k by dividing out as many powers of b as we can.

For polynomials $f(x)$ and $g(x)$ we write $f^n \| g$ to mean that f^n divides g in $\mathbb{Z}[x]$, but that no higher power of f divides g.

Lemma. Let $k \geq 1$. Then $b^{\lfloor k/2 \rfloor} \| p_k$.

Proof. Induction.

Define $q_k = p_k / b^{\lfloor \frac{k}{2} \rfloor}$ for all $k \geq 1$.

A simpler sequence still

Lemma. $q_1 = b + cx$, $q_2 = b + 2xc$, and for $k \geq 3$,

$$q_k = \begin{cases} q_{k-1} + xcq_{k-2} & k \text{ even}, \\ bq_{k-1} + xcq_{k-2} & k \text{ odd}. \end{cases}$$

A simpler sequence still

Lemma. $q_1 = b + cx$, $q_2 = b + 2xc$, and for $k \geq 3$,

$$q_k = \begin{cases} q_{k-1} + xcq_{k-2} & k \text{ even,} \\ bq_{k-1} + xcq_{k-2} & k \text{ odd.} \end{cases}$$

Proof. Induction.

A simpler sequence still

Lemma. $q_1 = b + cx$, $q_2 = b + 2xc$, and for $k \geq 3$,

$$q_k = \begin{cases} q_{k-1} + xcq_{k-2} & k \text{ even}, \\ bq_{k-1} + xcq_{k-2} & k \text{ odd}. \end{cases}$$

Proof. Induction.

Lemma. Let $k \geq 1$. Then $c^{\lceil \frac{k}{2} \rceil} \| q_k$.

A simpler sequence still

Lemma. $q_1 = b + cx$, $q_2 = b + 2xc$, and for $k \geq 3$,

$$q_k = \begin{cases} q_{k-1} + xcq_{k-2} & k \text{ even,} \\ bq_{k-1} + xcq_{k-2} & k \text{ odd.} \end{cases}$$

Proof. Induction.

Lemma. Let $k \geq 1$. Then $c^{\lceil \frac{k}{2} \rceil} \| q_k$.

Proof. Induction.

A simpler sequence still

Lemma. $q_1 = b + cx$, $q_2 = b + 2xc$, and for $k \geq 3$,

$$q_k = \begin{cases} q_{k-1} + xcq_{k-2} & k \text{ even,} \\ bq_{k-1} + xcq_{k-2} & k \text{ odd.} \end{cases}$$

Proof. Induction.

Lemma. Let $k \geq 1$. Then $c^{\lceil \frac{k}{2} \rceil} \| q_k$.

Proof. Induction.

Define $r_k = q_k / c^{\lceil \frac{k}{2} \rceil}$ for all $k \geq 1$.

At last...

Lemma. $r_1 = \frac{b}{c} + x$, $r_2 = \frac{b}{c} + 2x$, and for $k \geq 3$,

$$r_k = \begin{cases} r_{k-1} + x r_{k-2} & k \text{ even}, \\ \frac{b}{c} r_{k-1} + x r_{k-2} & k \text{ odd}. \end{cases}$$

At last...

Lemma. $r_1 = \frac{b}{c} + x$, $r_2 = \frac{b}{c} + 2x$, and for $k \geq 3$,

$$r_k = \begin{cases} r_{k-1} + x r_{k-2} & k \text{ even}, \\ \frac{b}{c} r_{k-1} + x r_{k-2} & k \text{ odd}. \end{cases}$$

Proof. Induction.

At last...

Lemma. $r_1 = \frac{b}{c} + x$, $r_2 = \frac{b}{c} + 2x$, and for $k \geq 3$,

$$r_k = \begin{cases} r_{k-1} + xr_{k-2} & k \text{ even}, \\ \frac{b}{c} r_{k-1} + xr_{k-2} & k \text{ odd}. \end{cases}$$

Proof. Induction.

Consequently,

$$\deg(r_k) = \begin{cases} k & k \text{ even}, \\ k+1 & k \text{ odd}. \end{cases}$$

At last...

Lemma. $r_1 = \frac{b}{c} + x$, $r_2 = \frac{b}{c} + 2x$, and for $k \geq 3$,

$$r_k = \begin{cases} r_{k-1} + x r_{k-2} & k \text{ even}, \\ \frac{b}{c} r_{k-1} + x r_{k-2} & k \text{ odd}. \end{cases}$$

Proof. Induction.

Consequently,

$$\deg(r_k) = \begin{cases} k & k \text{ even}, \\ k+1 & k \text{ odd}. \end{cases}$$

Thus when k is even, Proposition S applies, showing that every root of r_k (and thus of q_k and p_k) is real.

The case for $t \geq 3$

For more general t, nearly identical proofs apply, yielding the same result.

The case for $t \geq 3$

For more general t, nearly identical proofs apply, yielding the same result.

The only difference is in the complexity of the degree formulas and the inductive steps.

Generating examples

Let $G = \cup_{i=1}^{n} G_i$, $U = \cup_{i=1}^{n} U_i$, where $U_i \subseteq V(G_i)$. Let $H = \cup_{i=1}^{n} H_i$, where $H_i = G_i - U_i$. Thus $H = G - U$.

Suppose

Generating examples

Let $G = \cup_{i=1}^{n} G_i$, $U = \cup_{i=1}^{n} U_i$, where $U_i \subseteq V(G_i)$. Let $H = \cup_{i=1}^{n} H_i$, where $H_i = G_i - U_i$. Thus $H = G - U$.

Suppose

1. all roots of $I(G_i; n)$ are real and distinct, for $i = 1, ..., n$,

Generating examples

Let $G = \cup_{i=1}^{n} G_i$, $U = \cup_{i=1}^{n} U_i$, where $U_i \subseteq V(G_i)$. Let $H = \cup_{i=1}^{n} H_i$, where $H_i = G_i - U_i$. Thus $H = G - U$.

Suppose

1. all roots of $I(G_i; n)$ are real and distinct, for $i = 1, ..., n$,
2. there is a permutation π of $\{1, ..., n\}$ such that for all $i = 1, ..., k$ satisyfing $H_i \neq \emptyset$, $G_{\pi(i)} = H_i$, and

Generating examples

Let $G = \cup_{i=1}^{n} G_i$, $U = \cup_{i=1}^{n} U_i$, where $U_i \subseteq V(G_i)$. Let $H = \cup_{i=1}^{n} H_i$, where $H_i = G_i - U_i$. Thus $H = G - U$.

Suppose

1. all roots of $I(G_i; n)$ are real and distinct, for $i = 1, ..., n$,
2. there is a permutation π of $\{1, ..., n\}$ such that for all $i = 1, ..., n$ satisyfing $H_i \neq \emptyset$, $G_{\pi(i)} = H_i$, and
3. if $G = G' \cup H$, then $\deg(I(G'; x)) = 2$.

Generating examples, continued

Note that by (1) $b(x) = \prod I(G_i; x)$ and $c(x) = \prod I(H_i; x)$ both have only real roots.

Generating examples, continued

Note that by (1) $b(x) = \prod I(G_i; x)$ and $c(x) = \prod I(H_i; x)$ both have only real roots.

Moreover, the other conditions imply that $c|b$ in $\mathbb{Z}[x]$ and $\deg(b) = \deg(c) + 2$.

A more particular example

Let $\vec{m} = (m_1, m_2, ..., m_n)$ and $\vec{u} = (u_1, u_2, ..., u_n)$ be n-vectors of positive integers such that

A more particular example

Let $\vec{m} = (m_1, m_2, ..., m_n)$ and $\vec{u} = (u_1, u_2, ..., u_n)$ be n-vectors of positive integers such that

1. 0 appears exactly twice in $\vec{m} - \vec{u} = (m_1 - u_1, m_2 - u_2, ..., m_n - u_n)$ and

A more particular example

Let $\vec{m} = (m_1, m_2, ..., m_n)$ and $\vec{u} = (u_1, u_2, ..., u_n)$ be n-vectors of positive integers such that

1. 0 appears exactly twice in $\vec{m} - \vec{u} = (m_1 - u_1, m_2 - u_2, ..., m_n - u_n)$ and
2. each remaining value appears at most as many times as it does in $\vec{m}$.

A more particular example

Let $\vec{m} = (m_1, m_2, ..., m_n)$ and $\vec{u} = (u_1, u_2, ..., u_n)$ be n-vectors of positive integers such that

1. 0 appears exactly twice in $\vec{m} - \vec{u} = (m_1 - u_1, m_2 - u_2, ..., m_n - u_n)$ and
2. each remaining value appears at most as many times as it does in $\vec{m}$.

Then letting $G_i = K_{m_i}$ and $|U_i| = u_i$ yields G and U meeting the above criteria, so $P(t, k)\nabla(G, U)$ is log-concave in these cases.

The next steps

Question. What if $\deg(b) > \deg(c) + 2$?

The next steps

Question. What if $\deg(b) > \deg(c) + 2$?

Example. Consider $p_k(x) = I(P(2,k)\nabla(N_s, V(N_s)); x)$, the polynomials corresponding to s-regular caterpillars.

The next steps

Question. What if $\deg(b) > \deg(c) + 2$?

Example. Consider $p_k(x) = I(P(2,k)\nabla(N_s, V(N_s)); x)$, the polynomials corresponding to s-regular caterpillars.

A simple recurrence appears:

$$p_k = (x+1)^s(p_{k-1} + xp_{k-2}),$$

in which $\deg(p_k) - \deg(p_{k-1}) = s$.

The next steps

Question. What if $\deg(b) > \deg(c) + 2$?

Example. Consider $p_k(x) = I(P(2,k)\nabla(N_s, V(N_s)); x)$, the polynomials corresponding to s-regular caterpillars.

A simple recurrence appears:

$$p_k = (x+1)^s(p_{k-1} + xp_{k-2}),$$

in which $\deg(p_k) - \deg(p_{k-1}) = s$.

For $s \geq 3$, we need analogues of Theorem 0 and Proposition S .

References

BAHLS, P., “Logarithmic concavity of inpendence polynomials of path-like graphs,” preprint, 2009.

BAHLS, P. and SALAZAR, N., “Properties of certain nonuniform recursive trees,” Symmetry and unimodality of independence polynomials of generalized paths,” submitted to *Australas. J. Combin.*, 2009.

LEVIT, V.E. and MANDRESCU, E., “A family of graphs whose independence polynomials are both palindromic and unimodal,” *Carpathian J. Math.* **23** (2007) no. 1-2, 108–116.

ZHU, Z.-F., “The unimodality of independence polynomials of some graphs,” *Australas. J. Combin.* **38** (2007) 27–33.