Recent Results in Secure Domination

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Outline

1. Define secure dominating set & secure domination number
2. Examples
3. Known bounds
4. Improved bound (short proof)
5. Trees with equality in bounds
6. Open problems

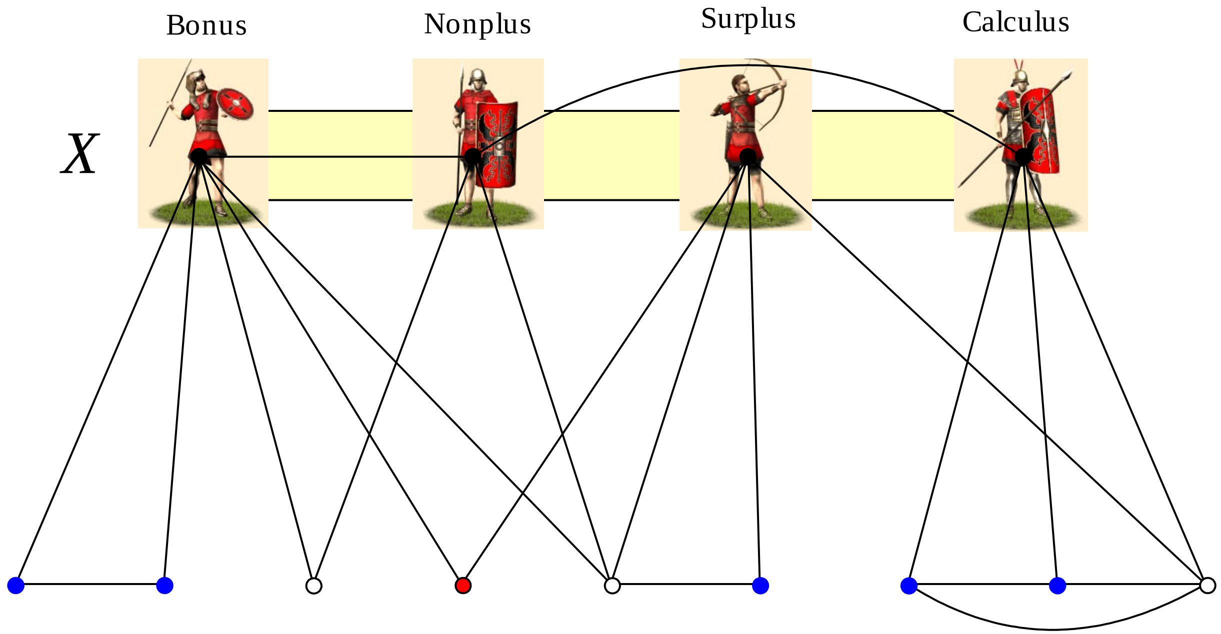
Secure domination:

Graph protection strategy – single guards on vertices of G

- ♦ form a dominating set of G
- ♦ if guard on v moves along edge to protect unguarded u, then
- ♦ new guards form dominating set.

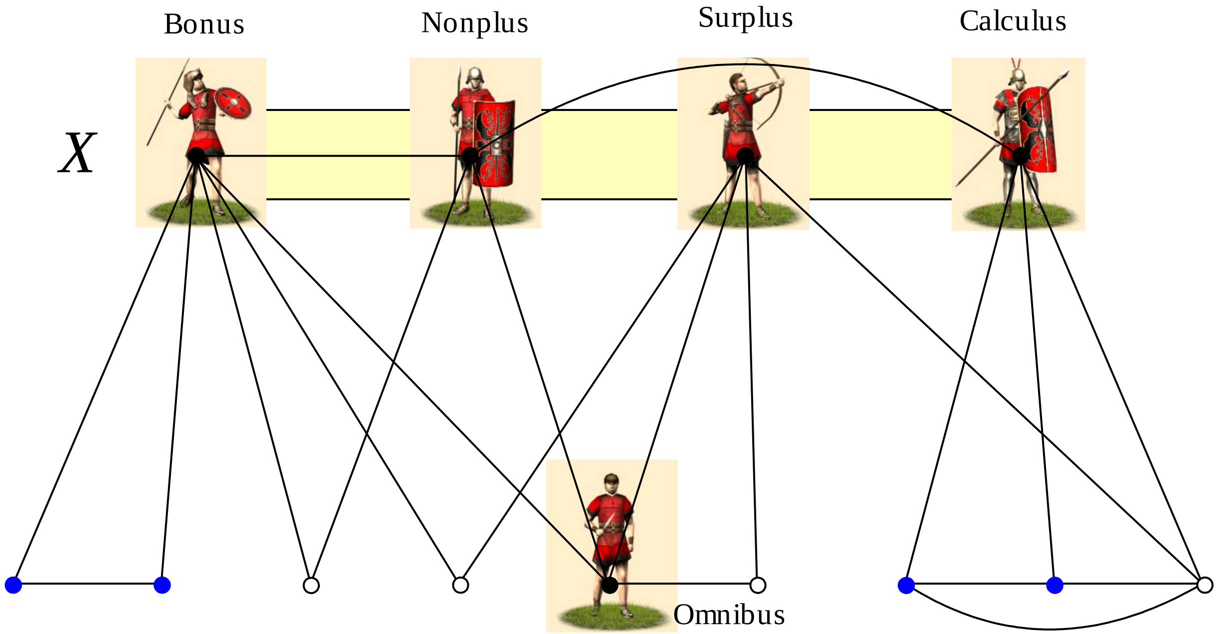
Secure dominating set:

Vertex set X with the property that
for each $u \in V - X$
there exists $v \in X$ adjacent to u such that
$(X - \{v\}) \cup \{u\}$ is dominating.



Vertex $u \in V - X$ is X-protected by x if and only if $\langle\{u, x\} \cup \mathrm{epn}(x, X)\rangle$ is complete. (1)

X is a secure dominating set if and only if for each $u \in V - X$ there exists $x \in X$ such that (1) holds.



Known Bounds

Sharp Lower Bound (Cockayne, Favaron, Mynhardt, 2004)

If G is K_t-free with maximum degree Δ, then

$$\gamma_s(G) \geq \frac{n(2\Delta - 2t + 5)}{(\Delta+1)^2 - (t-1)(t-2)}.$$

Upper Bounds

1. (CFM 2004) If G is a claw-free graph with n vertices and minimum degree δ, then

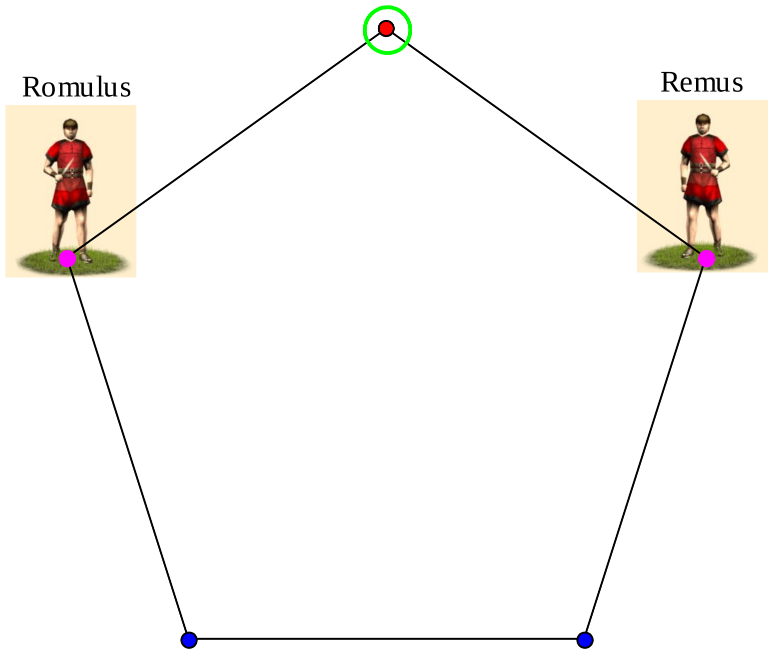
$$(i)\ \ \gamma_s(G) \ \leq \ 2\gamma(G)$$

$$(ii)\ \ \gamma_s(G) \ \leq \ \begin{cases} \beta(G) & \text{if } G \text{ is also } C_5\text{-free} \\ \frac{3}{2}\beta(G) & \text{otherwise} \end{cases}$$

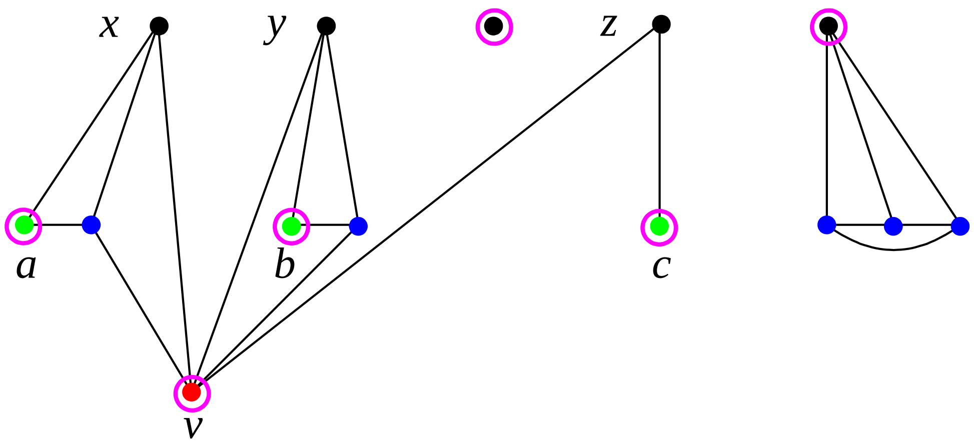
$$(iii)\ \ \gamma_s(G) \ \leq \ \frac{3n}{\delta+3}.$$

Upper Bounds

2. Trivial: $\gamma_s(G) \leq \theta(G)$ (clique covering number)
3. Very easy: $\gamma_s(G) \leq 2\beta(G) - 1$
4. Improvement: If G is C_5-free, then $\gamma_s(G) \leq \beta(G)$.



β-set X:



Suppose v is not X-protected and consider all neighbours of v in X.

Then v is nonadjacent to some private neighbour (green) of each of these vertices.

Suppose two of these private neighbours are adjacent (red edge).

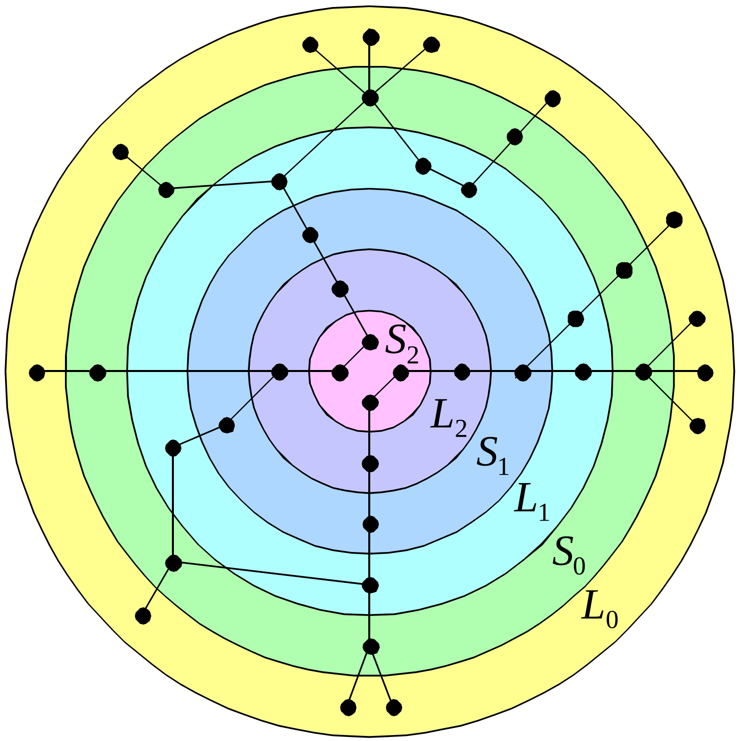
Oops! Then we have an induced C_5 (red edges)!

Thus the private neighbours (green), the non-neighbours of v in X, and v

form a larger independent set – oops again.

Trees with $\gamma_s = \beta$

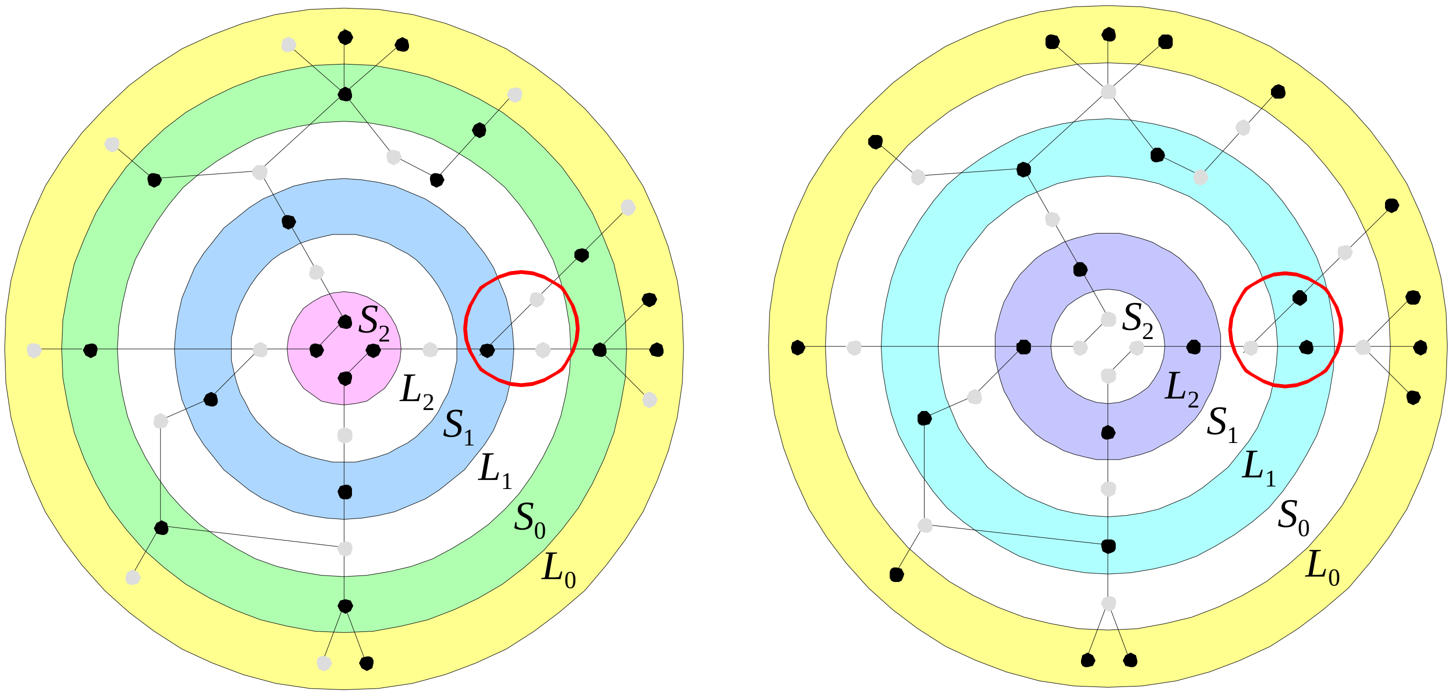
Draw T on a large slice of onion.
Leaves L_0 on the outer ring, support vertices S_0 on the next ring.
Leaves L_1 and isolated vertices I_1 of $T-(L_0 \cup S_0)$ on the next ring.
Support vertices S_1 of $T-(L_0 \cup S_0)$ on the next ring, etc.



Trees with $\gamma_s = \beta$

Then $\gamma_s(T) = \beta(T)$ if and only if the following conditions hold:

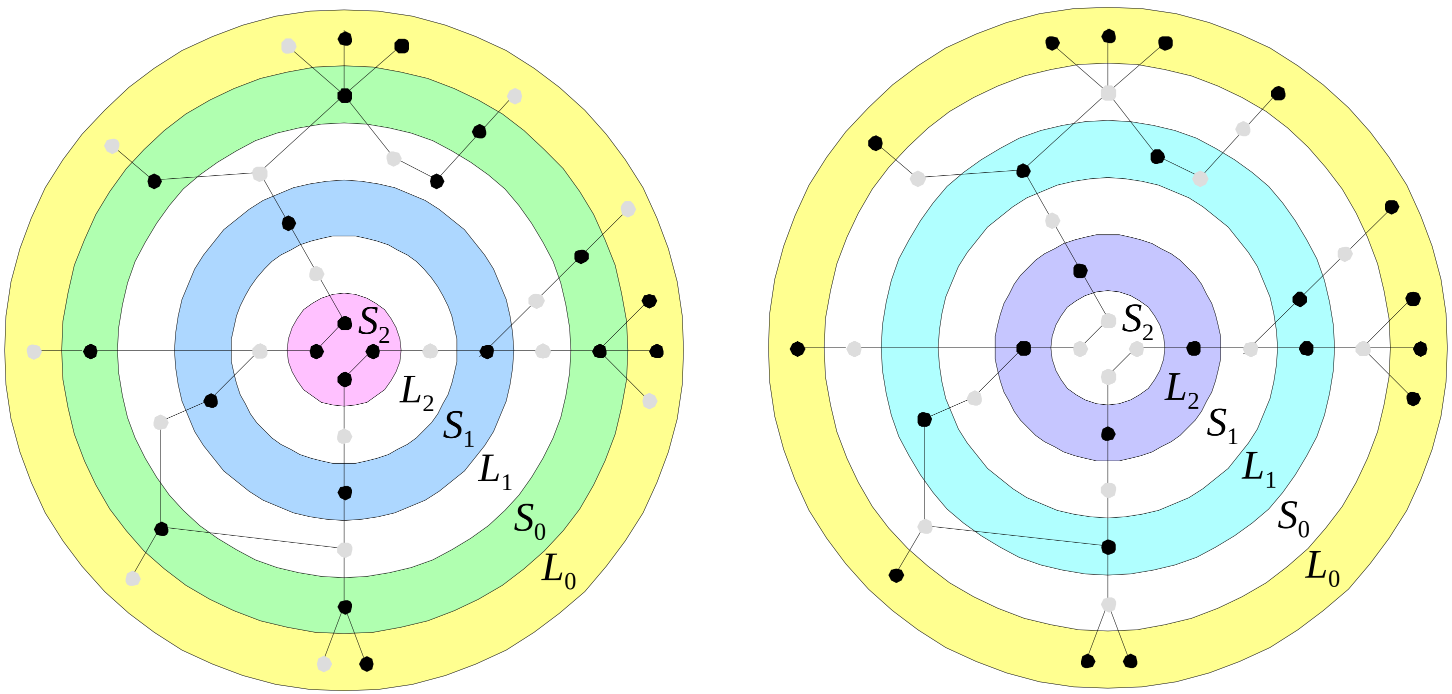
1. Six onion rings are enough ($L_i = S_i = \emptyset$ for $i \geq 3$).
2. There are no isolated vertices in $\langle L_2 \cup S_2 \rangle$.
3. For $i = 1, 2$, each vertex in S_i is adjacent to exactly one vertex in L_i.



Trees with $\gamma_s = \beta$

Then $\gamma_s(T) = \beta(T)$ if and only if the following conditions hold:

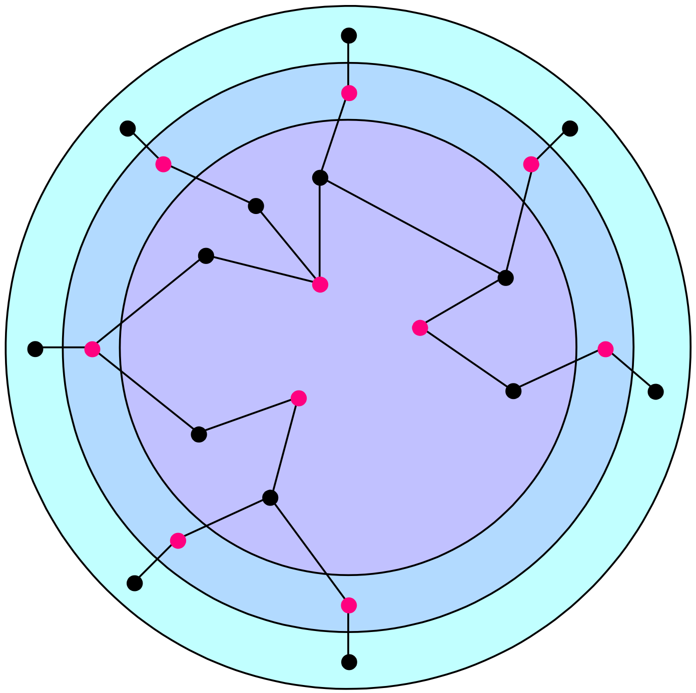
4. If $\deg v \geq 3$ for $v \in L_2 \geq 3$, then $uv \in E(T)$ for $u \in L_2$ with $\deg u = 2$.
5. Condition (a) about $v \in S_0$ adjacent to more than one vertex in $T - (L_0 \cup S_0)$.
6. Condition (b) about $v \in S_0$ adjacent to more than one vertex in $T - (L_0 \cup S_0)$.



Trees with $\gamma_s = \gamma$

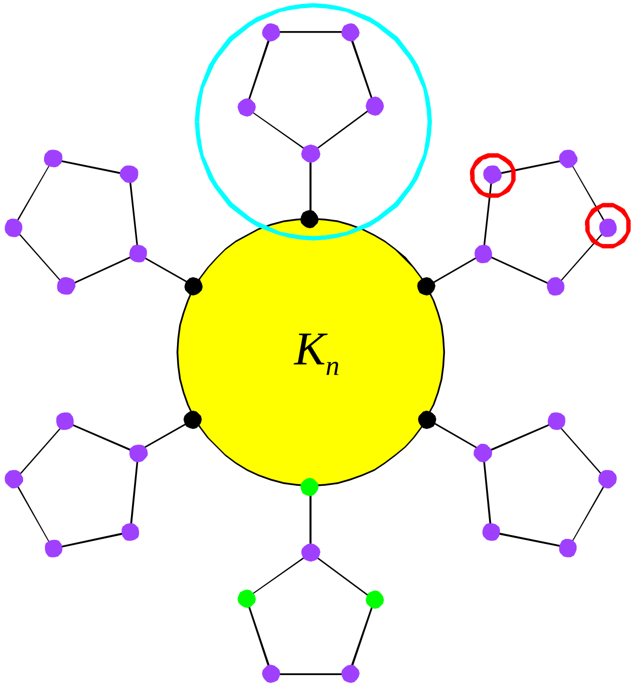
$\gamma_s(T) = \gamma(T)$ if and only if the following conditions hold:

1. No vertex is adjacent to more than one leaf.
2. $T - (L \cup S)$ has an efficient DS – centres of disjoint stars of order at least three.
3. Centres not adjacent to any other vertices; leaves of distinct stars may be adjacent.



Known connected graphs with $\gamma_s = \frac{3}{2}\beta$: Only graphs with $\beta = 2$.

Connected graphs with $\gamma_s = \frac{3}{2}\beta - \varepsilon$:

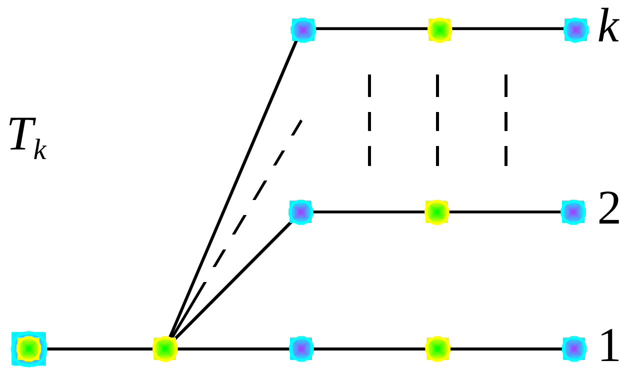


G_n

$$\beta(G_n) = 2n + 1,\ \gamma_s(G_n) = 3n$$

Questions about γ_s and β

1. Is it true that $\gamma_s \leq \frac{3\beta}{2}$ for all graphs? (Not just claw-free)
2. Does equality hold for connected graphs only if $\beta = 2$? (Claw-free or otherwise)
3. $\beta(T) - \gamma_s(T)$ may be arbitrary (≥ 0) for trees, but what is the ratio $\gamma_s(T)/\beta(T)$?



$\gamma_s(T_k) = k + 2$, $\beta(T_k) = 2k + 1$; so $\min\{\gamma_s(T)/\beta(T)\} \leq \frac{1}{2} + \varepsilon$.

4. Is it true that $\gamma_s(T) > \frac{1}{2}\beta(T)$ for all trees?

Questions about γ_s and γ

Recall: If G is claw-free, then $\gamma_s \leq 2\gamma$.
Claw-free is necessary if G has leaves.
But, for example, $K_{m,n}$ has many claws if m, n large enough, and for $m, n \geq 4$,

$$\gamma_s(K_{m,n}) = 4 = 2\gamma(K_{m,n}).$$

Problems:

1. Find weaker conditions under which $\gamma_s \leq 2\gamma$.
2. For example, when is $\gamma_s(T) \leq 2\gamma(T)$ if T is a tree? $\gamma_s(T) = 2\gamma(T)$?
3. What is the relationship between γ_s and γ if G is $K_{1,t}$-free?
4. Can we bound γ_s/γ by restricting $\delta(G)$?
5. Find other classes of graphs for which $\gamma_s = \gamma$.

Complexity

And of course, design algorithms to compute γ_s for

1. trees
2. bipartite graphs
3. general graphs, etc.

and determine the various complexities.