Conflict and Tolerance in the Representations of Graphs

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Paradigm for
conflict-tolerance representation of a graph G

- each vertex is assigned a *size*

- each vertex is assigned a *tolerance*.

- A method for combining the sizes of two vertices v and w to produce a single combined size.

- A method for combining the tolerances of two vertices v and w to produce a single combined tolerance.

- A *conflict rule*, based solely on the combined size and combined tolerance, which provides a way of deciding if a *conflict* exists between v and w.

Persistency Condition

If a particular choice of sizes
and tolerances produces a conflict
and if the sizes are increased **and/or**
the tolerances decreased,
then the conflict should persist.

REPRESENTATION

The assignment yields a *representation* iff
the pairs of vertices in conflict
are precisely the edges of G.

Conflict-Tolerance MODEL

A set of choices for size, tolerance,
combination, and conflict
will be called a *conflict-tolerance model*.

Standard Choices

- Sizes are *numeric* or *sets*.

- If sets, they come from some (universal) *host* set H.

- Sets may be *structured* or *unstructured*.

- Tolerances may be *numeric* or *configurational*.

- A model is *intersectional* if it uses intersection in its conflict rule.

- An intersectional model is *simple* iff nonempty intersection of sizes produces a conflict.

More Standard Choices

- If a conflict exists iff the combined sizes is at least some threshold value, then this is a *constant tolerance* model.

- A model is *universal* iff every graph is representable.

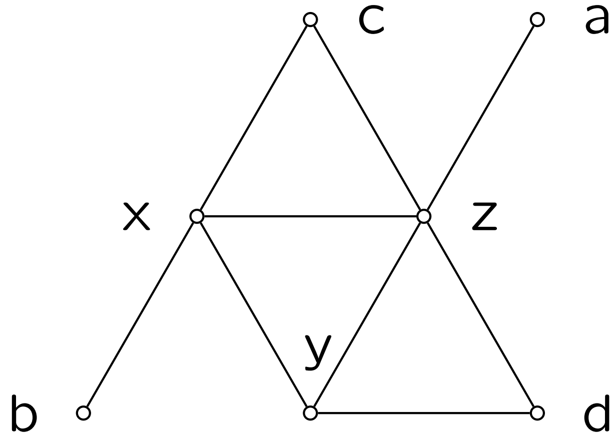
Central question about universal models

- How small can the representation of a graph be made?

Central questions about non-universal models

- Which graphs are representable?

- How easily can one recognize the representable graphs?



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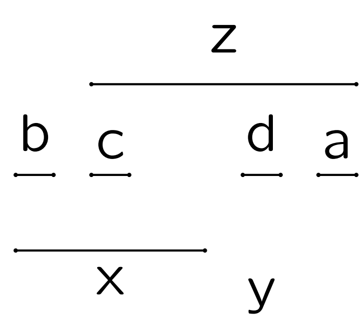


A set model that is not persistent. Take host tree $H = T$ and assign to each vertex v of G a path $S(v)$. Two vertices are in conflict iff one of the paths representing them contains at least a quarter of the nodes of the other. That is, v and w should be adjacent iff

$$\frac{1}{4}\min(|S(v)|, S(w)|) \leq |S(v) \cap S(w)|.$$

A purely numeric model. (Golumbic and Jamison)

Each vertex v is assigned a rank(size) $r_v \in \mathbb{R}^+$ and a tolerance $t_v \in \mathbb{R}^+$.
Ranks are combined by taking their product and tolerances are combined by taking their sum.

Conflict rule:

$$t_v + t_w \leq r_v r_w$$

If t_v does not vary with v, then we have a constant tolerance model.

A model that mixes numeric and set. (Golumbic and Monma)

The size $S(v)$ of a vertex is an interval on the real line.

The tolerance of a vertex v is a positive real number t_v.

Conflict rule:

$$vw \in E \Leftrightarrow \min(t_v, t_w) \leq |S(v)| \cap |S(w)|.$$

Some unstructured configurational models

Each vertex v is assigned an unstructured set $S(v)$ as size. Sizes are combined by intersection, so the model is intersectional. Instead of tolerances, we have a family $\mathscr{F}$ of *tolerated configurations*. The conflict rule looks like this: v and w are in conflict iff their combined sizes is *not* a tolerated configuration.

Non-persistent Conflict Rules.

Source	$vw \in E$ iff
Eaton and Grable	$\lvert S(v) \cap S(w)\rvert$ is odd
West	$S(v) \setminus S(w) \neq \emptyset$ and $S(w) \setminus S(v) \neq \emptyset$
This article	$\lvert S(v) \cap S(w)\rvert \leq \frac{1}{2}\lvert S(v) \cup S(w)\rvert$

Two structured configurational models.

In both models, the host H is an arbitrary graph and $S(v)$ is a subtree of H.

In the first model, two subtrees conflict iff their intersection contains a claw $K_{1,3}$.

Note that the tolerated configurations are just the linear subgraphs – unions of paths.
This is clearly an intersectional model.

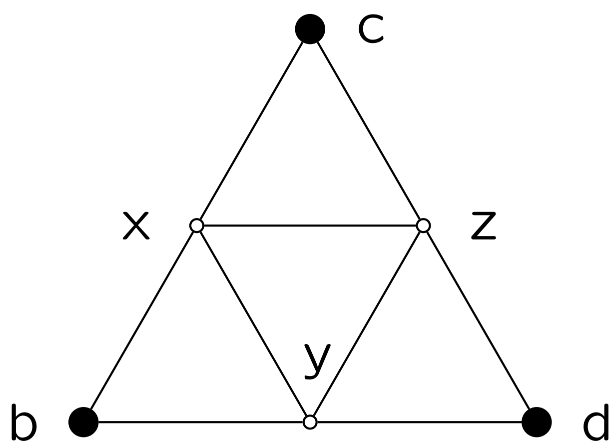
In the second model union instead of intersection is used. Two subtrees conflict in this model iff their union contains a cycle.

Simple Set Intersection Models.

An *intersection graph* is a graph representable in a model in which the sizes are sets (structured or unstructured) and having constant tolerance 1 with the conflict rule:

$$vw \in E \iff S(v) \cap S(w) \neq \emptyset.$$

$S(v)$	Class Name
arbitrary sets	intersection graphs
intervals on the real line	interval graphs
subtrees of a tree	choral graphs
subpaths of a tree	path (VPT) graphs
arcs on a circle	circular arc graphs
unions of k-intervals	k-interval graphs



3-Sun

A *chord* of a cycle C in a graph G is an edge of G that joins two vertices of C but is not an edge of C.

A graph is *chordal* iff every cycle of length 4 or more has a chord.

A set of three vertices v_1, v_2, v_3 in a graph $G = (V, E)$ is an *asteroidal triple* (AT) iff there are paths P_1, P_2, P_3 such that for each i, P_i contains the vertices v_{i-1} and v_{i+1} and v_i is not adjacent to any vertex of P_i.

Erdos, Goodman, and Posa (1966)

Every graph is the intersection graph of some family of (unstructured) sets.

Lekkerkerker and Boland (1962)

A graph is an interval graph iff it is chordal and contains no asteroidal triples.

Gavril (1974) and Walter(1978)

A graph is an intersection graph of subtrees of a tree iff it is chordal.

Constant-tolerance Unstructured Set Intersection Models

We now replace the simple conflict rule $vw \in E \Longleftrightarrow S(v) \cap S(w) \neq \emptyset$ by the rule

$$vw \in E \Longleftrightarrow t \leq |S(v) \cap S(w)|$$

where t is some fixed.

Let $\theta_t(G)$ be the minimum h such that the graph G has a t-intersection representation on a host with h points. It is well-known that $\theta_1(G)$ is the smallest number of cliques required to cover the edges of G. Erdos, Goodman and Posa proved that $\theta_1(G)$ is maximized for a given order n of a graph G by the complete balanced bipartite graph. Eaton has shown that $\theta_t(G)$ FAILS to be maximized by the complete balanced bipartite graph for all $t \geq 2$ and n sufficiently large.

Constant-tolerance Structured Set Intersection Models.

For positive integer parameters h, s, and t, Jamison and Mulder defined the class $[h, s, t]$ to consist of all graphs having a representation in a host tree with maximum degree h by subtrees with maximum degree s and with constant tolerance t. A particularly nagging open question is

$$\text{Conj:} \quad [h, s, t] \subseteq [h, s, t'] \quad \text{if } t \leq t'$$

This was proved by Jamison and Mulder for $t = 2, 3, 4$ for all trees and by Eaton and Faubert for all t when the host is a caterpillar.

Jamison and Mulder showed that every graph belongs to $[d, d, t]$ for some t if $d \geq 3$. Let $t_d(G)$ denote the minimum t such that $G \in [d, d, t]$. Eaton et al. have studied $t_3(G)$. It would be of interest to study the behavior of $t_d(G)$ as $d \to \infty$.

Eaton and her coworkers have considered representations in host trees whose structure is restricted but not by degree. Namely, they have looked for representations of cycles in

- caterpillars and

- 3-asters (subdivisions of $K_{1,3}$).

The numeric case. (Golumbic and Jamison)

In this model, each vertex of a graph is assigned a positive rank (the size) and a positive tolerance. When the combined rank is at least the combined tolerance, there is a conflict and an edge appears in the conflict graph.

In this approach, the combinations of ranks and tolerances depend on two commutative functions ϕ and ρ on the first quadrant. A graph $G = (V, E)$ is a (ϕ, ρ)-*graph* iff there is an assignment $v \to t_v$ of **positive** *tolerances* to the vertices and an assignment $v \to r_v$ of **positive** *ranks* to the vertices such that

$$xy \in E \iff \phi(t_x, t_y) \leq \rho(r_x, r_y) \qquad (1)$$

The class of all (ϕ, ρ)-graphs is denoted by $[\phi, \rho]$. Standard choices for φ and ρ are combinations of such elementary functions as min, max, and sum.

$\rho =$ / $\phi =$	min	max	sum	prod
min	Th	coIn	TT	TT
max	In	Th	coTT	coTT
sum	coTT	TT	Th	SP
prod	coTT	TT	coSP	Th

Th = Threshold; In = Interval

$$mix_\alpha(x, y) = \alpha \max(x, y) + (1 - \alpha) \min(x, y)$$

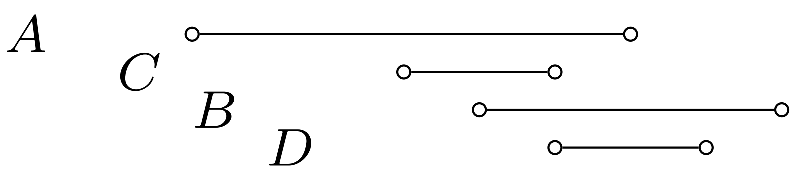
	Property of $[mix_\alpha, mix_\beta]$	Known range of validity
	General Assumption $\longrightarrow$	$\alpha \geq 0$ and $\beta \geq 0$
	Contains coTT	$1 > \alpha > 0$ and $\beta = 0$
	Contains all trees and all interval graphs	$\alpha > 1$ and $\beta = 0$
	Equals $co[mix_\beta, mix_\alpha]$	All $\alpha \geq 0$ or $\beta \geq 0$
	Equals $[mix_{1-\beta}, mix_{1-\alpha}]$	$1 \geq \alpha$ and $1 \geq \beta$
	Contains all threshold graphs	$\alpha, \beta \geq 0$
	Contained in chordal graphs	$1 \geq \alpha \geq \beta$
	Contained in split graphs	$0 \leq \alpha = \beta \leq 1$
	$A(4,4,4)$-free	$1 \geq \alpha \geq \beta$
	Contains all unit interval graphs	$\alpha > \beta$
	Closed under disjoint unions	$\alpha > \beta$
	Contains all complete	$\alpha < \beta \leq 1$
	Equals threshold graphs	$\alpha = \beta = 0, \frac{1}{2}, 1$
	Equals interval graphs	$\alpha = 1, \beta = 0$

Configurational models

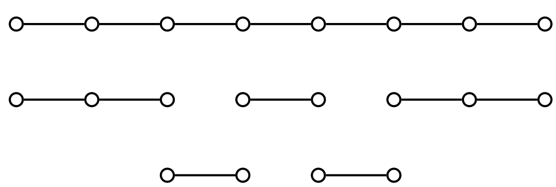
In this model studied by West, sets are assigned to the vertices of a graph, and two vertices are adjacent iff their sets *overlap* – that is, iff their intersection is nonempty and neither set is contained in the other. The representable graphs are called *overlap graphs*. In this model, disjoint sets are tolerated and comparable sets are tolerated.

Eaton and Grable have studied the configurational model in which an intersection of odd cardinality leads to a conflict whereas an intersection of even cardinality is tolerated. These two models are *unstructured set* models and are not persistent.

Let us now add structure by requiring that the representing sets be intervals on the real line. For the odd intersection model, the representation must be discretized. That is, the host is a long path and the intervals are replaced by subpaths.



C_4 as an overlap interval graph.



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Constant Tolerance Structured Set Model

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Purely Numeric Model