COLORED PROBLEMS } IN GRAPHS
PARTITION PARAMETERS }

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COLORED-DOMINATION

(Suk Seo + PJS)
(CONG. NUM. 167)

COLORED-INDEPENDENCE

+ PJS, CONG. NUM.

COLOR-COMPLETE-CYCLES

GAME THEORETIC:

C
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ALTERNATE MOVES WITH AN OPPONENT:

INDEPENDENCE $i \le \beta_{com}, \beta^-_{com} \le \beta$

ENCLAVELESS $\psi \le \Psi_{com}, \Psi^-_{com} \le \Psi$

J. B. PHILLIPS + PJS
→ GTNNY, 2000
→ CONG. NUM, 2002

ACQUISITION
YAN WANG

C
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OPPONENT INITIALLY SETS SOME CONDITIONS:

⋮

$$\pi : V(G) \xrightarrow{\text{ONTO}} \{1, 2, 3, \ldots, t\}$$

$$V_i = \pi^{-1}(i)$$

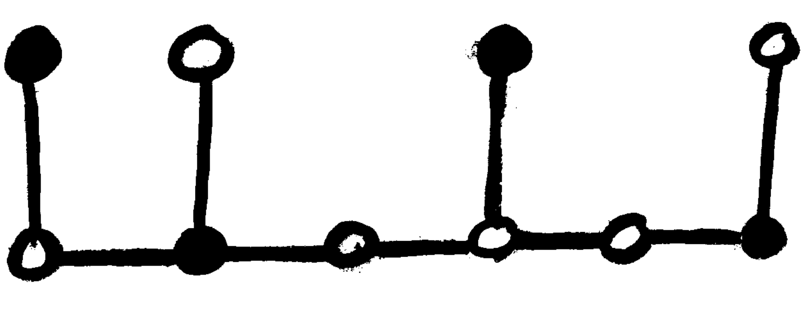
$$\mathcal{S} = \{V_1, V_2, \ldots, V_t\}$$

$$V(G) = V_1 \,\dot{\cup}\, V_2 \,\dot{\cup} \ldots \dot{\cup}\, V_t$$

$$\gamma(G;\mathcal{S}) = \text{MIN}\left\{ \left|\bigcup_{i \in I} V_i\right| : I \subseteq \{1, 2, \ldots, t\},\ N\left[\bigcup_{i \in I} V_i\right] = V(G) \right\}$$

$$= \gamma(G;\pi)$$

$$\boxed{\gamma_{cpl}(G)} = \text{MAX}\{\gamma(G;\mathcal{S}) : |V_i| \leq 2\ \forall\, V_i \in \mathcal{S}\}$$



THM. $\rightarrow \gamma(G) \leq \gamma_{cpl}(G) \leq 2 \cdot \gamma(G)$.

$\rightarrow \gamma(G) \leq \gamma(G; \mathcal{S})$

$\leq \gamma(G) \cdot \mathrm{MAX}\{|V_i| : V_i \in \mathcal{S}\}$

OBS. $\mathcal{S} = \{V_1, V_2, V_3, \ldots, V_t\}$

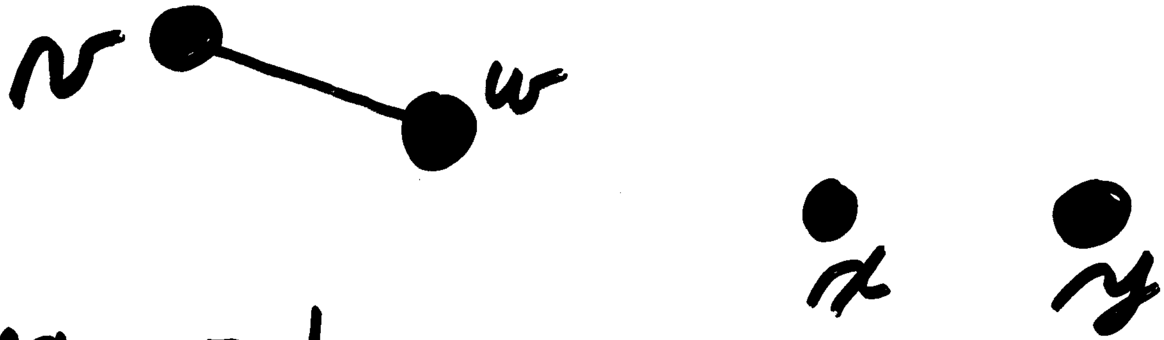
$\mathcal{S}^* = \{V_1 \cup V_2, V_3, \ldots, V_t\}$

$\Rightarrow \gamma(G; \mathcal{S}) \leq \gamma(G; \mathcal{S}^*)$

COR. $\exists\, \mathcal{S} = \{V_1, V_2, \ldots, V_t\}$, $t = \lceil n/2 \rceil$

WITH $\gamma_{cpl}(G) = \gamma(G; \mathcal{S})$

and $|V_i| \leq 2$.



$\deg v = 1$

$\exists\, \mathcal{S} = \{V_1, V_2, \ldots, V_t\} \ni$

$\gamma_{cpl}(G) = \gamma(G; \mathcal{S})$

and $V_1 = \{w, v\}$

<u>CASE I</u>: $\gamma_{cpl}(G) = \gamma(G, \mathcal{S})$

$\mathcal{S} = \{V_1, V_2, V_3, \ldots, V_t\}$

$V_1 = \{w, x\}$ $\quad V_2 = \{v, y\}$

LET $\mathcal{S}^* = \{\{w, v\}, \{x, y\}, V_3, \ldots, V_t\}$

D^* A COUPLED-DOM. SET FOR $\mathcal{S}^*$

$\Rightarrow \{w, v\} \subseteq D^*$

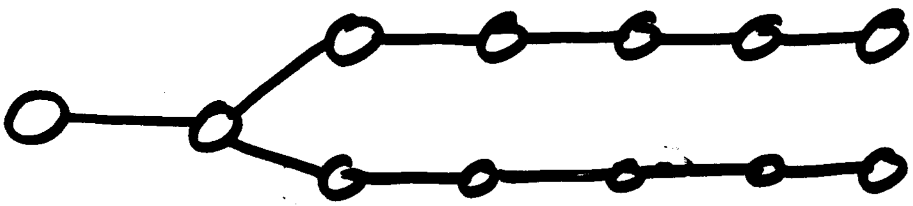
IF $\{x, y\} \subseteq D^*$, $\gamma(G, \mathcal{S}) \le |D^*| = \gamma(G; \mathcal{S}^*)$

IF $\{x, y\} \not\subseteq D^*$, LET $D = D^* - v \cup \{x\}$

$\gamma(G; \mathcal{S}) \le |D| = |D^*| = \gamma(G; \mathcal{S}^*)$

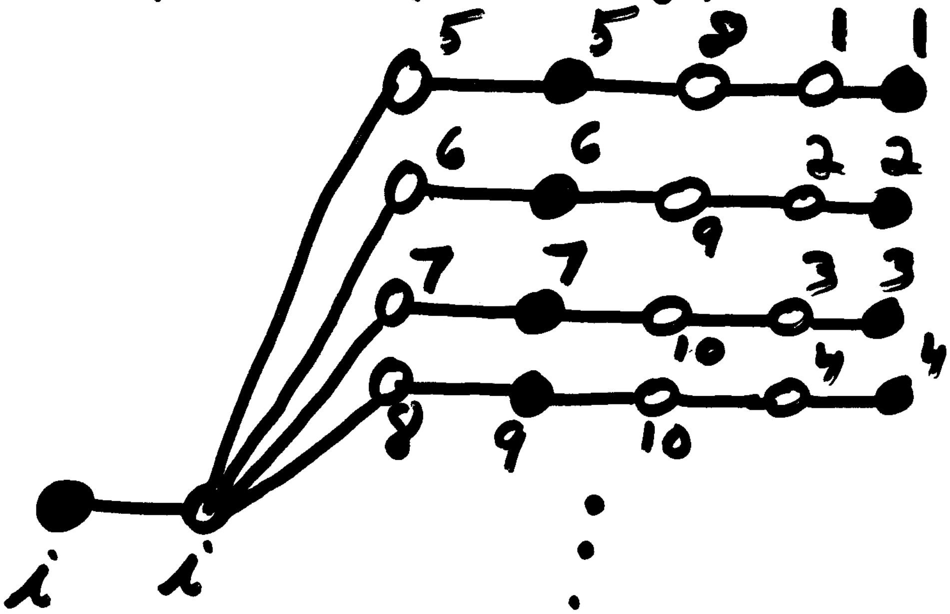
$$1 \le \frac{\gamma_{cpl}(G)}{\gamma(G)} \le 2$$

<u>QUESTION</u>: MIN $\dfrac{\gamma_{cpl}(T_n)}{\gamma(T_n)} = ?$



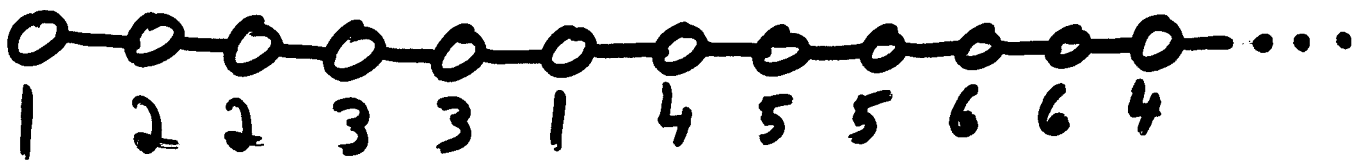
T_{12}

$$\gamma_{cpl}(T_{12}) / \gamma(T_{12}) = 8/5$$



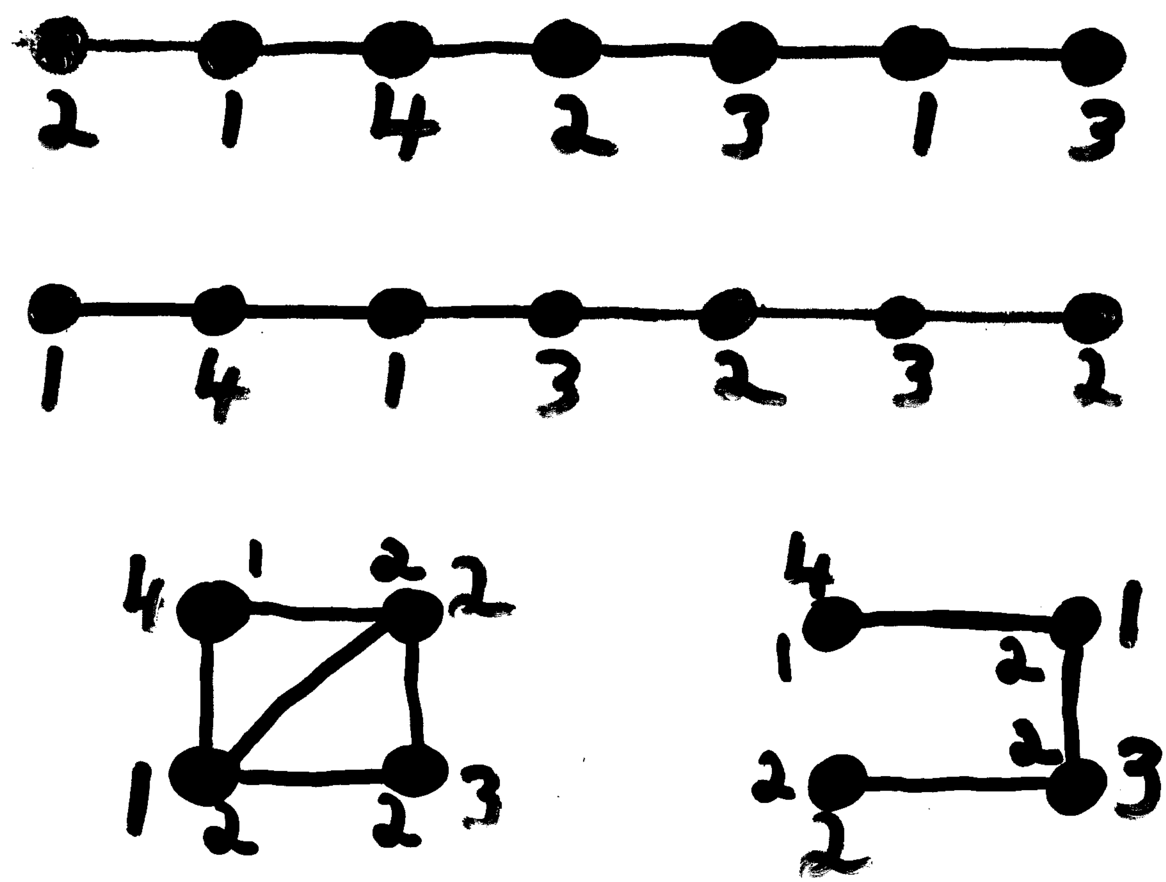
T_{20j+2}

$$\frac{\gamma_{cpl}(T_{20j+2})}{\gamma(T_{20j+2})} = \frac{14j+2}{8j+1}$$



$$\gamma_{cpl}(P_{6t}) = 4t = 2 \cdot \gamma(P_{6t})$$

COLORED INDEPENDENCE



$$\mathcal{S} = \{S_1, S_2, \ldots, S_t\}$$

$$V(G) = S_1 \dot\cup S_2 \dot\cup \ldots \dot\cup S_t \quad , \quad S_t \neq \phi$$

$$I \subseteq \{1, 2, \ldots, t\}$$

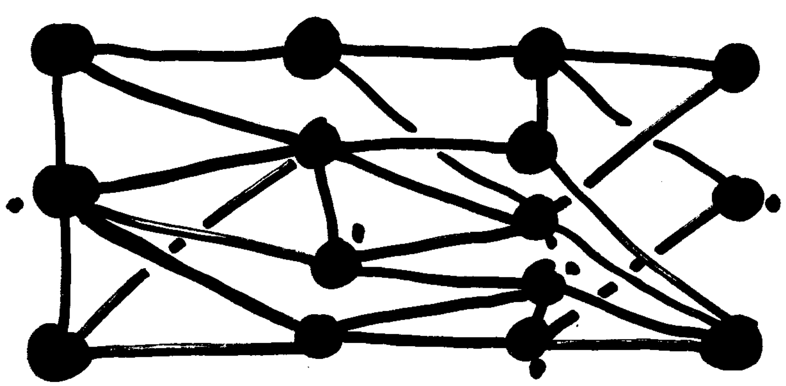
$$\mathcal{S}_I = \bigcup_{i \in I} S_i \quad \subseteq \quad V(G)$$

$$\beta(G; \mathcal{S}) = MAX\{|\mathcal{S}_I| : \mathcal{S}_I \text{ is independent}\}$$

$$\alpha(G; \mathcal{S}) = MIN\{|\mathcal{S}_I| : \mathcal{S}_I \text{ is a vertex cover}\}$$

$$\gamma(G; \mathcal{S}) = MIN\{|\mathcal{S}_I| : \mathcal{S}_I \text{ dominates } V(G)\}$$

$$\Psi(G; \mathcal{S}) = MAX\{|\mathcal{S}_I| : \mathcal{S}_I \text{ is enclaveless}\}$$



THM. $\alpha(G)+\beta(G)=|V(G)|$ (GALLAI, 1959)

$\gamma(G)+\Psi(G)=|V(G)|$ (PJS, 1977)

(MATRIX) COMPLEMENTATION THEOREM (PJS, 1996-98)

THM. $\forall$ partition $\mathcal{S}$ of $V(G)$

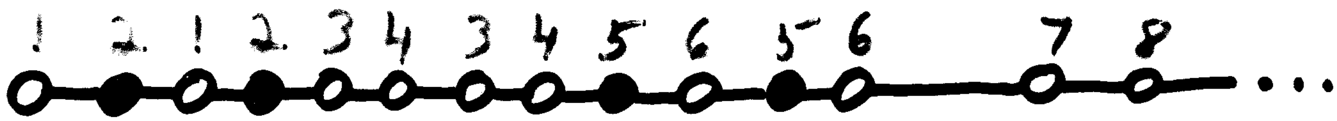
$$\alpha(G;\mathcal{S})+\beta(G;\mathcal{S})=|V(G)|$$

$$\gamma(G;\mathcal{S})+\Psi(G;\mathcal{S})=|V(G)|$$

$\ldots ir(G) \le \gamma(G) \le i(G) \le \ldots \le \beta(G) \le \Gamma(G) \le IR(G)$..

LOWER INDEPENDENCE

$i(G;\mathcal{S})$ = MINIMUM CARDINALITY OF AN $\mathcal{S}$-INDEPENDENT SET THAT IS MAXIMAL.



$$i(G;\mathcal{S}) = i(P_{12k};\mathcal{S}) = 4k = n/3$$

$$\gamma(G;\mathcal{S}) = \gamma(P_{12k};\mathcal{S}) = 6k = n/2$$

$$\beta(G;\mathcal{S}) = \text{MAX}\{|\mathcal{S}_I| : \mathcal{S}_I \text{ is independent}\}$$

$\mathcal{S} = \{S_1, S_2, \ldots, S_t\}$ is PROPER if each S_i is independent.

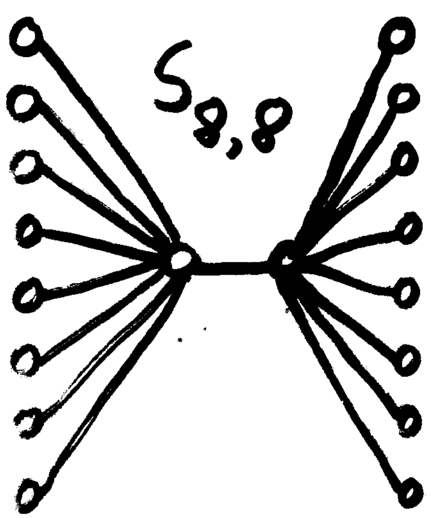
$$\beta_{PRT}(G) = \text{MIN}\{\beta(G;\mathcal{S}) : \mathcal{S} \text{ is proper}\}$$

$$\beta_{PRT(k)}(G) = \text{MIN}\{\beta(G;\mathcal{S}) : \mathcal{S} \text{ is proper and } |S_i| \le k \ \forall S_i \in \mathcal{S}\}$$

$$\beta_{cpl}(G) = \beta_{PRT(2)}(G)$$

__THM.__ $\beta(G) = \beta_{PRT(1)}(G) \geq \beta_{PRT(2)}(G) \geq \ldots \geq \beta_{PRT(k)}(G) \geq \beta_{PRT(k+1)}(G) \geq \ldots \geq \beta_{PRT}(G)$

$S_{8,8}$

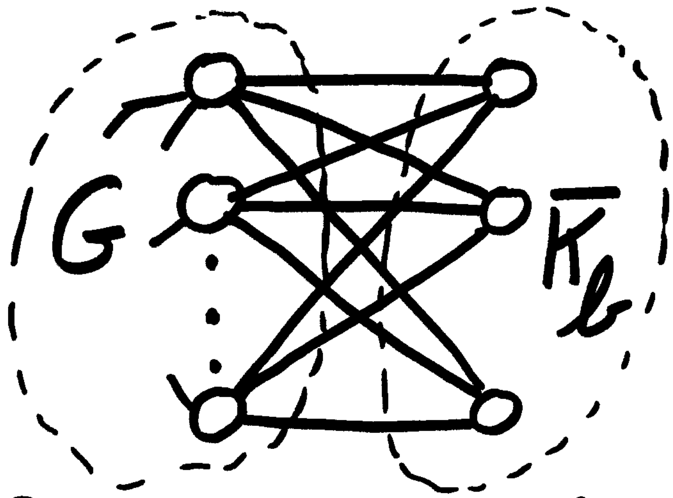


$n = 18$
$\beta = n-2 = 16$

$\beta_{cpl} =$

$\beta_{PRT} =$

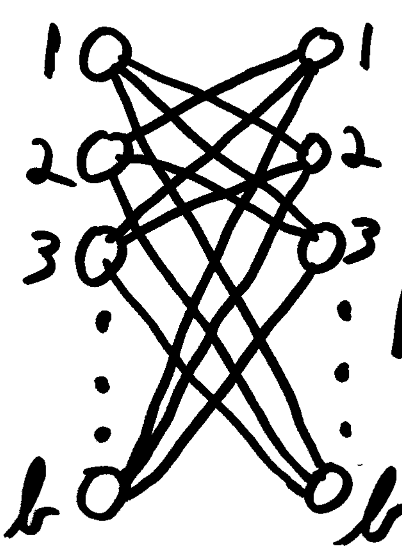
OBS. $\beta(G) \geq 2 \Rightarrow \beta_{PRT}(G) \geq 2.$



PROP. $\beta(G) \leq b \Rightarrow \beta(G+\overline{K}_b)$
$= \beta_{PRT}(G+\overline{K}_b) = b$

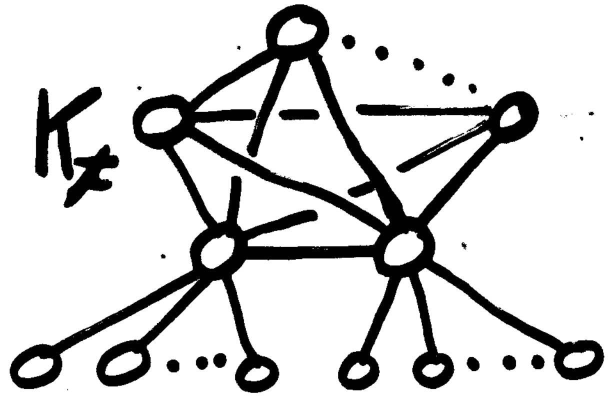
COR. $\beta(K_{n_1,n_2,\ldots,n_t}) = \beta_{PRT}(K_{n_1,n_2,\ldots,n_t})$
$= n_t$ WHERE $n_t \geq n_i, 1 \leq i \leq t.$

$\beta(K_{b,b}) = \beta_{PRT(2)}(K_{b,b}) = \ldots = \beta_{PRT}(K_{b,b}) = b.$



$\beta(K_{b,b}-M) = b$

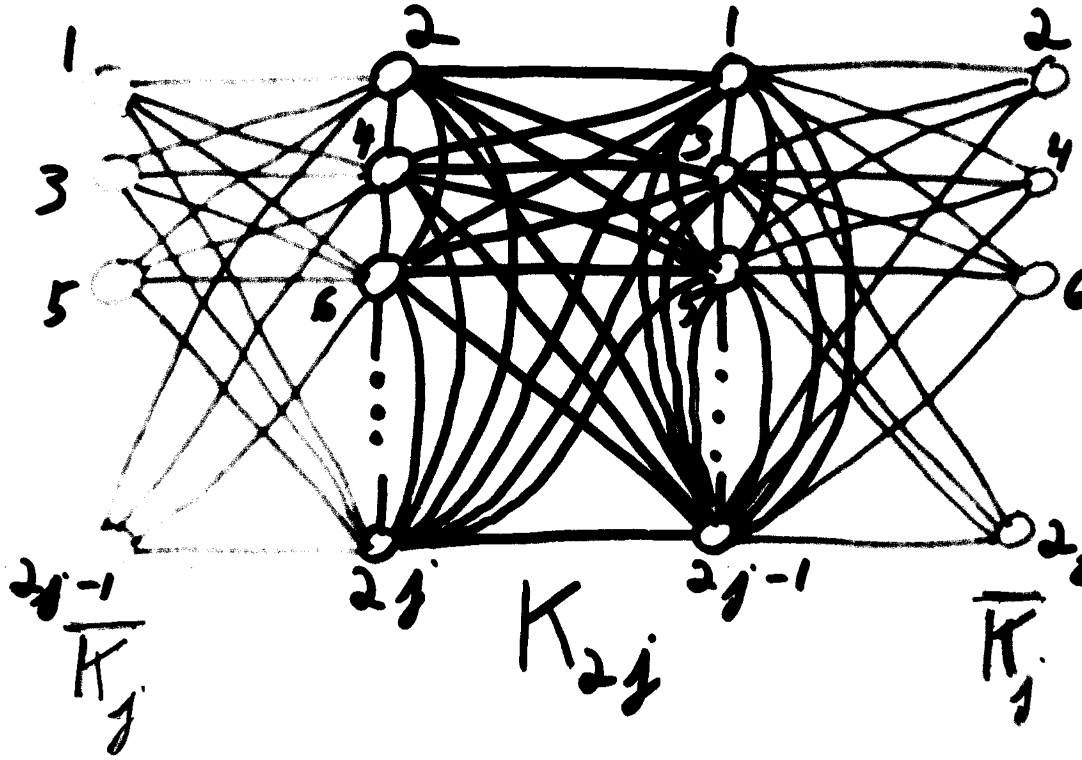
$\beta_{PRT(2)}(K_{b,b}-M) = \ldots = \beta_{PRT}(K_{b,b}-M)$
$= 2$



$\beta = n - t + 1$

$\beta_{cpl} = n - 2t + 2$

$\beta_{PRT} = \left\lceil \frac{n}{t} \right\rceil$



$\beta = 2j = n/2$

$i = j + 1$

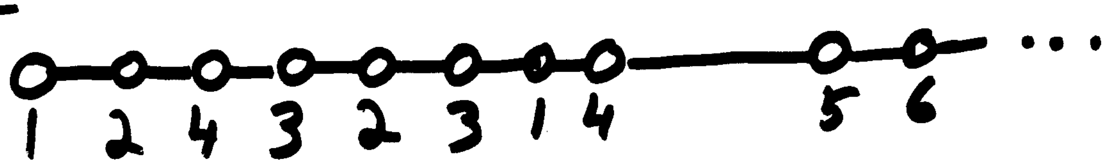
$\beta_{PRT(2)}(G) = \cdots = \beta_{PRT^*}(G) = 2$

PATHS $\quad \beta(P_n) = \lceil n/2 \rceil$

n	$8j$	$8j+1$ $8j+2$	$8j+3$ $\cdots$ $8j+6$	$8j+7$
$\beta_{PRT(2)}(P_n)$	$2j$	$2j+1$	$2j+2$	$2j+3$

$$\beta_{CPL}(P_{8j}) = 2j = n/4$$

Pf.



$$\beta_{CPL}(P_{8j}) \le \beta(P_{8j}; \mathcal{S}) = 2j = \frac{n}{4}$$

FOR ANY 2-PARTITION $\mathcal{S}$ of $V(P_{8j})$:

ITERATE: choose $v \ni \deg v \le 1$

PUT v IN S

DELETE v and any $w \in S_i$ with $S_i \cap (N(v))$

⋮

$|S| \ge 2j$. $\quad \beta(P_{8j}; \mathcal{S}) \ge 2j = \frac{n}{4}$

$\beta_{PRT(k)}(P_n)$: Given an S.

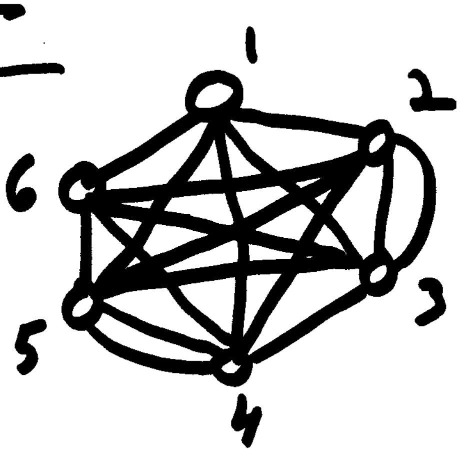
ITERATE: choose an endpoint v

Put the $j \le k$ vertices of this color in S

Delete the at most $j + (2j-1)k$ "color adjacent" vertices

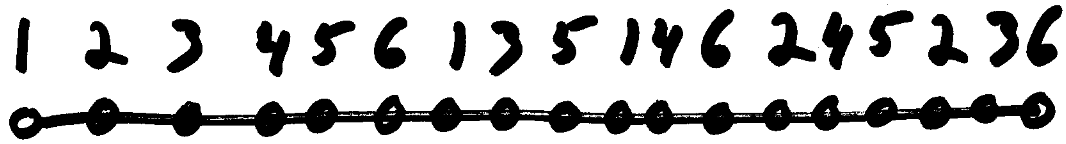
$$\frac{j}{j+(2j-1)k} \ge \frac{k}{2k^2} = \frac{1}{2k}$$

$$\Rightarrow \beta_{PRT(k)}(P_n) \ge \frac{n}{2k}$$



$$E(K_{2k}^{\#}) = E(K_{2k}) \cup \{\{2,3\},\{4,5\},\ldots,\{2k-2,2k-1\}\}$$

USE EULERIAN TRAIL



$$k=3: \beta_{PRT(3)}(P_{18j}) \le 3j = \frac{n}{2\cdot k}$$

$$\longrightarrow \beta_{PRT(k)}(P_{2k^2 j}) \le kj = n/2k$$

THM. $\beta_{PRT(k)}(P_{2k^2 j}) = kj$

$$= \frac{1}{2k} \cdot n$$

THM. $\beta_{PRT}(P_{2k^2}) = k = \frac{\sqrt{2}}{2} n^{1/2}.$