Symmetric Chains of Subsets and Necklaces

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Matching “parentheses” in a bit string

1 0 0 1 1 0 0 1 1 1 0 0 0 0 1 0

Matching "parentheses" in a bit string

1 0 0 1 1 0 0 1 1 1 0 0 0 0 1 0

unmatched 0's and unmatched 1's

1 * * * * * * * * 1 0 0 0 * * 0

Matching "parentheses" in a bit string

1 0 0 1 1 0 0 1 1 1 0 0 0 0 1 0

unmatched 0's and unmatched 1's

1 * * * * * * * * 1 0 0 0 * * 0

Unmatched 0's are always to the right of all unmatched 1's.

Changing the first unmatched 0 to 1 or the last unmatched 1 to 0 does not change matching.

Growing chains in the Boolean lattice $\mathcal{B}_n$

1100 0110 0101 1010 0011 1001

Growing chains in the Boolean lattice $\mathcal{B}_n$

1110	0111		1011		1101
1100	0110	0101	1010	0011	1001

Growing chains in the Boolean lattice $\mathcal{B}_n$

1111					
1110	0111		1011		1101
1100	0110	0101	1010	0011	1001

Growing chains in the Boolean lattice $\mathcal{B}_n$

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0000					

Gives a symmetric chain decomposition for every n [Greene-Kleitman 1976].

Pattern-avoiding permutations

Permutation $\pi_1\pi_2\ldots\pi_n$ *avoids* pattern "123" if there is no $i < j < k$ s.t. $\pi_i < \pi_j < \pi_k$.

Amazing result [Simion & Schmidt 1985]: The number of permutations that avoid 123 is independent of "123".

Same holds true for permutations of a multiset $1^{a_1}2^{a_2}\ldots n^{a_n}$ ([SW 2005], extending [Albert et al 2001])

Multiset permutations of $1^{a_1}2^{a_2}\ldots n^{a_n}$ avoiding 123

The number is the same for every permutation of $a_1, a_2, \ldots a_n$ [Albert et al 2001]

Bijection - suffices to show for swap of adjacent a_i:

$$1^{a_1}\ldots i^{a_i}(i+1)^{a_{i+1}}\ldots n^{a_n} \quad \leftrightarrow \quad 1^{a_1}\ldots i^{a_{i+1}}(i+1)^{a_i}\ldots n^{a_n}$$

Use Greene Kleitman [SW2005].

Example (assume $a_i > a_{i+1}$).

$$S = (1^2)(2^1)(3^1)(4^5)(5^2)(6^7)(7^1) \quad \to \quad T = (1^2)(2^1)(3^1)(4^2)(5^5)(6^7)(7^1)$$

Start with a string $x \in S$

```
7 5 6 6 4 6 6 4 6 6 4 6 5 3 2 4 1 1 4
```

Replace i by '(' and $i+1$ by ')'

```
7 ) 6 6 ( 6 6 ( 6 6 ( 6 ) 3 2 ( 1 1 (
```

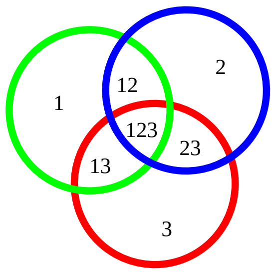
Match parentheses in the usual way. Change the leftmost $a_i - a_{i+1}$ unmatched left parentheses to right parentheses.

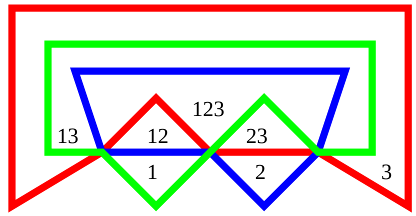
```
7 ) 6 6 ) 6 6 ) 6 6 ( 6 ) 3 2 ) 1 1 (
```

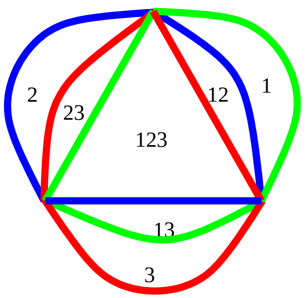
Change ')' back to $i+1$ and change '(' back to i.

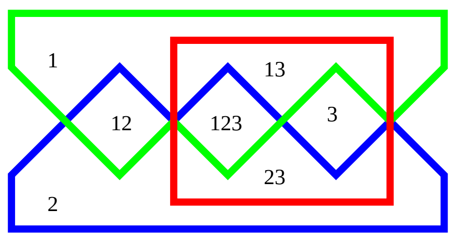
```
7 5 6 6 5 6 6 5 6 6 4 6 5 3 2 5 1 1 4
```

Venn Diagram for n sets









Dual is a graph whose vertices are the elements of $\mathcal{B}_n$.

Venn diagram for n sets: collection of n simple closed curves in the plane, $\{\Theta_1, \Theta_2, \ldots, \Theta_n\}$, such that for each $S \subseteq \{1, 2, \ldots, n\}$ the region

$$\bigcap_{i \in S} \mathrm{int}(\Theta_i) \quad \cap \quad \bigcap_{i \notin S} \mathrm{ext}(\Theta_i)$$

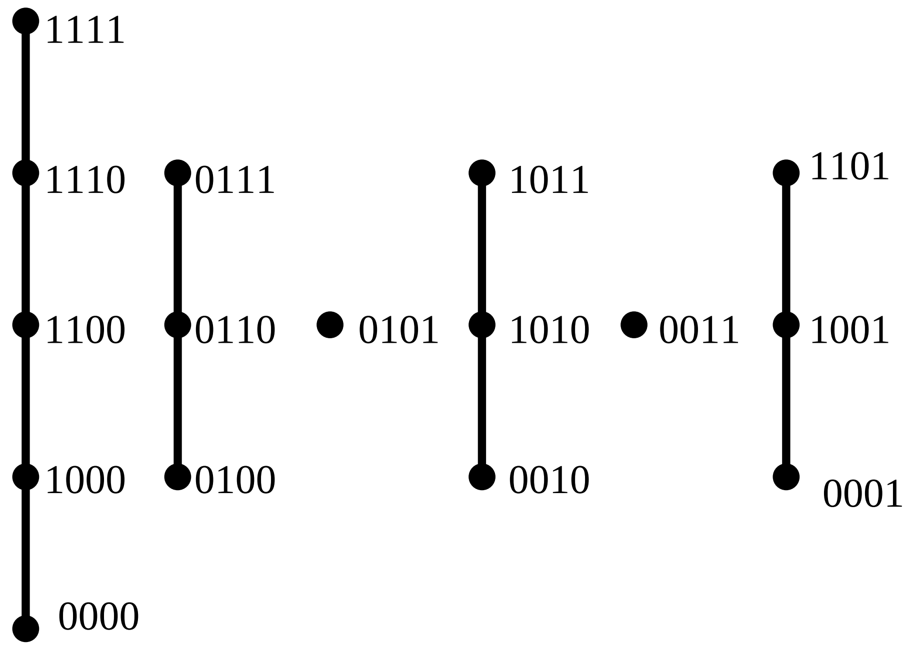
is nonempty and connected.

Back to $\mathcal{B}_n$

1111					
1110	0111		1011		1101
1100	0110	0101	1010	0011	1001
1000	0100		0010		0001
0000					

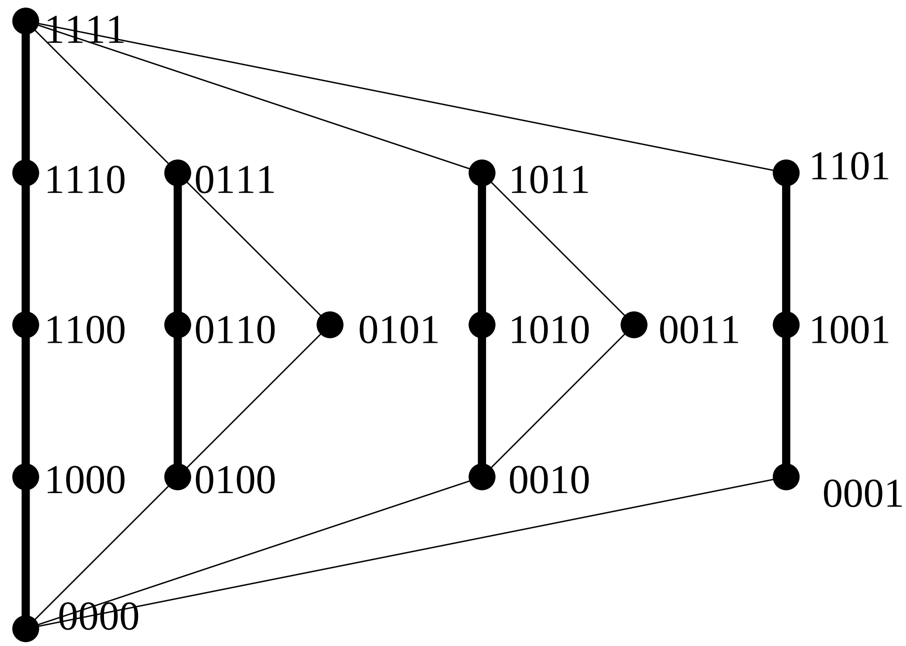
Start with Greene-Kleitman chains.

Back to $\mathcal{B}_n$



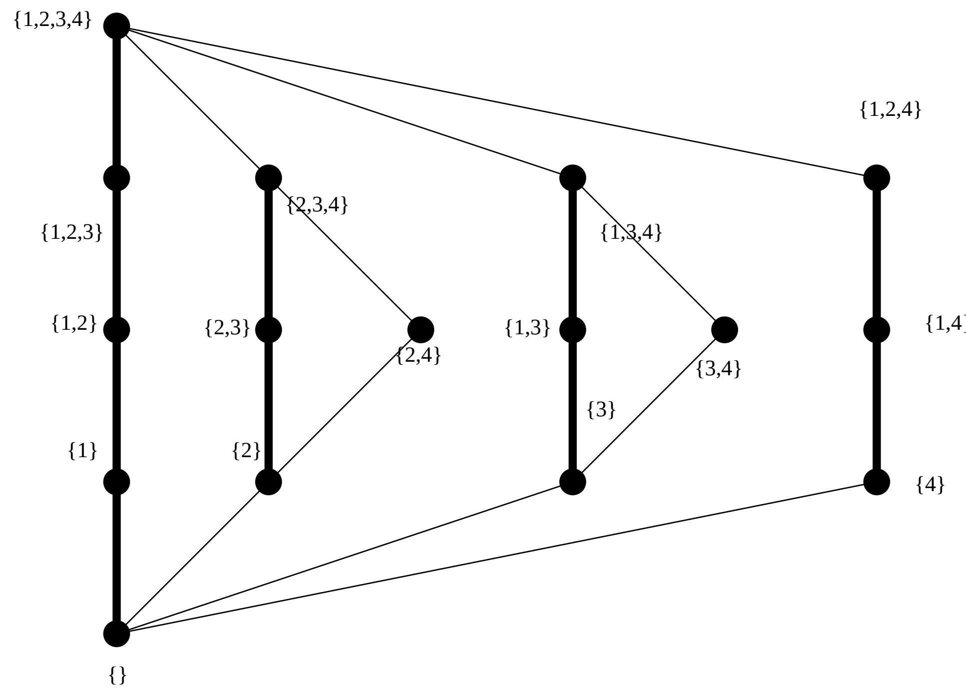
Associate a graph.

Back to $\mathcal{B}_n$



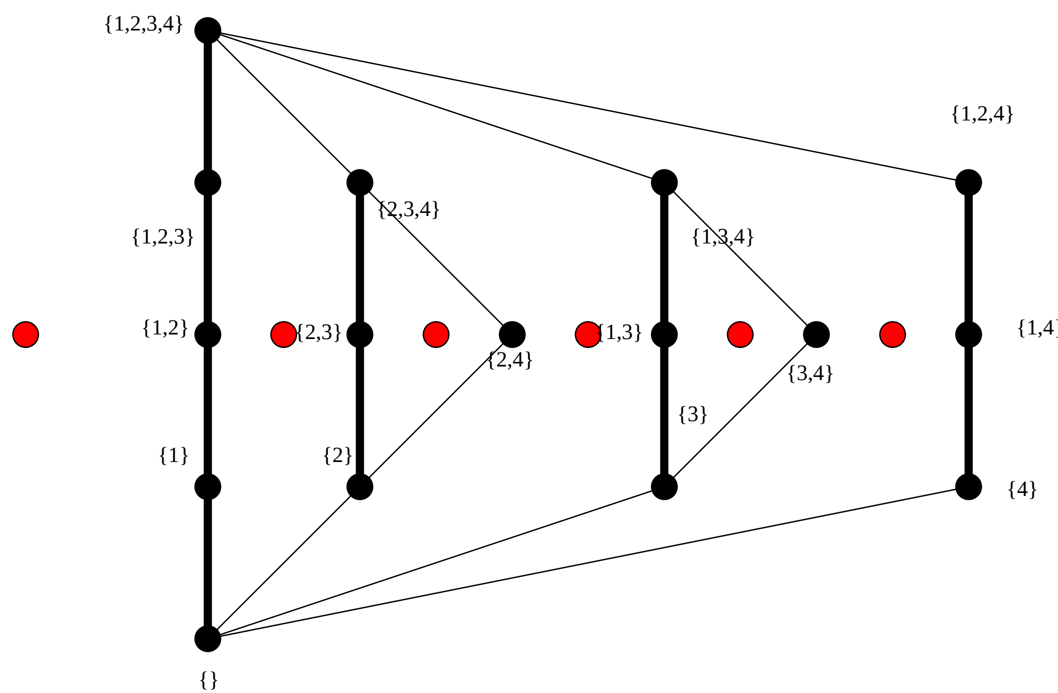
Add “chain cover edges”.

Back to $\mathcal{B}_n$



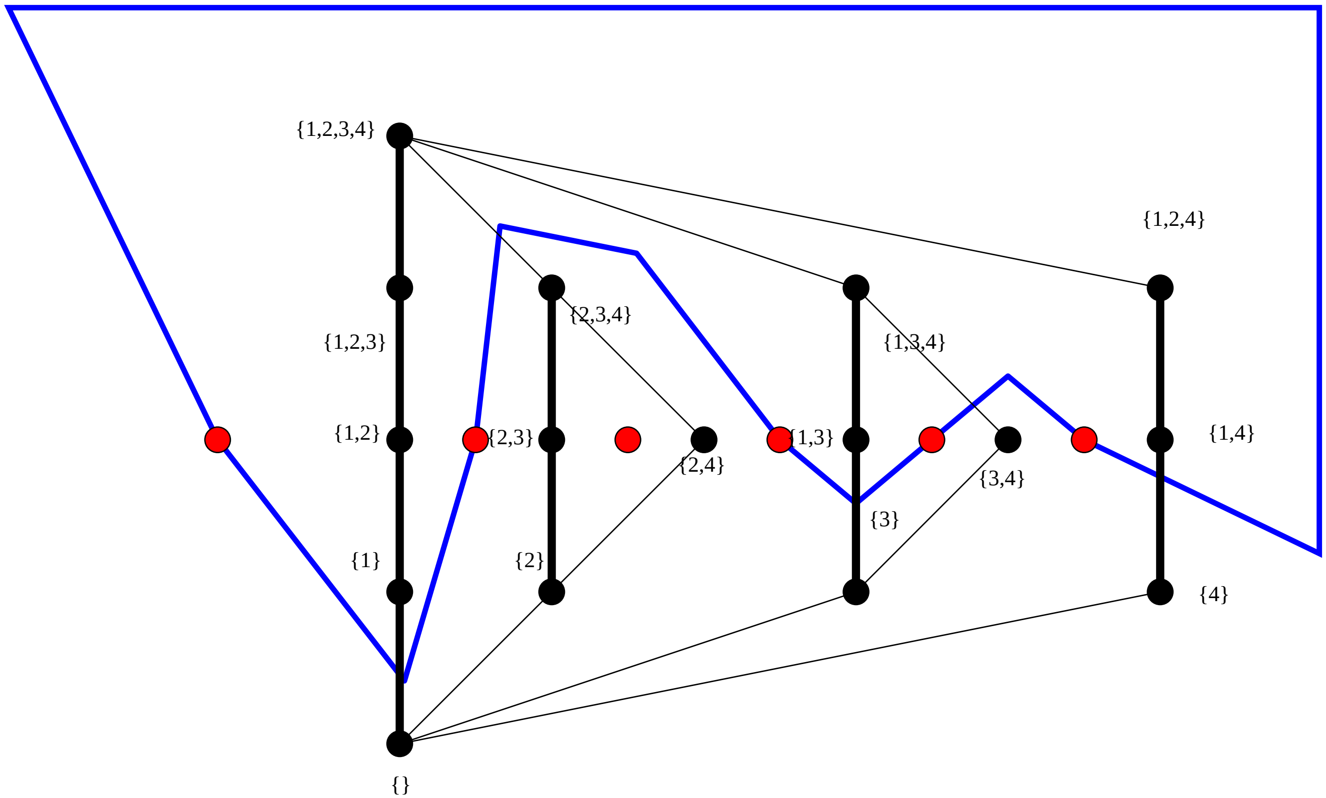
Convert from bit string to set notation.

Back to $\mathcal{B}_n$



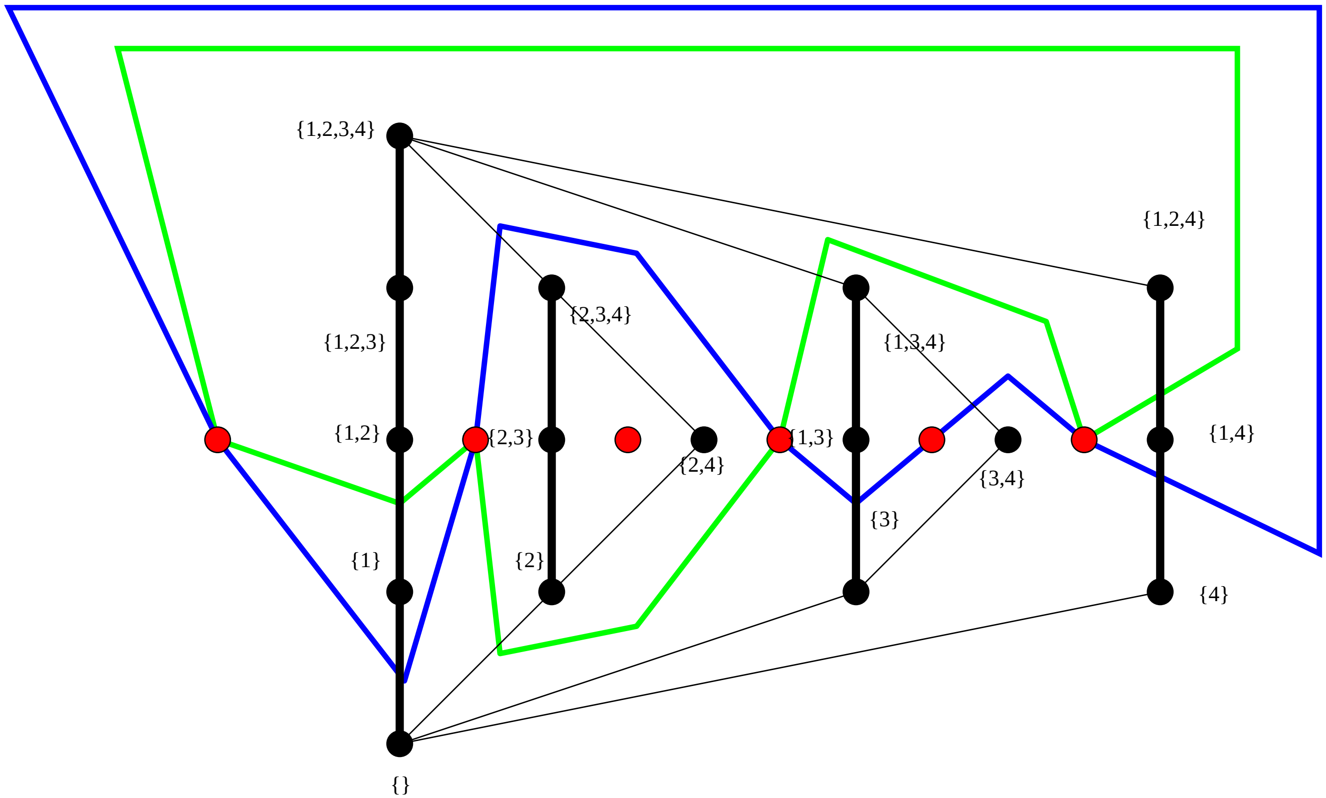
Take dual.

Back to $\mathcal{B}_n$



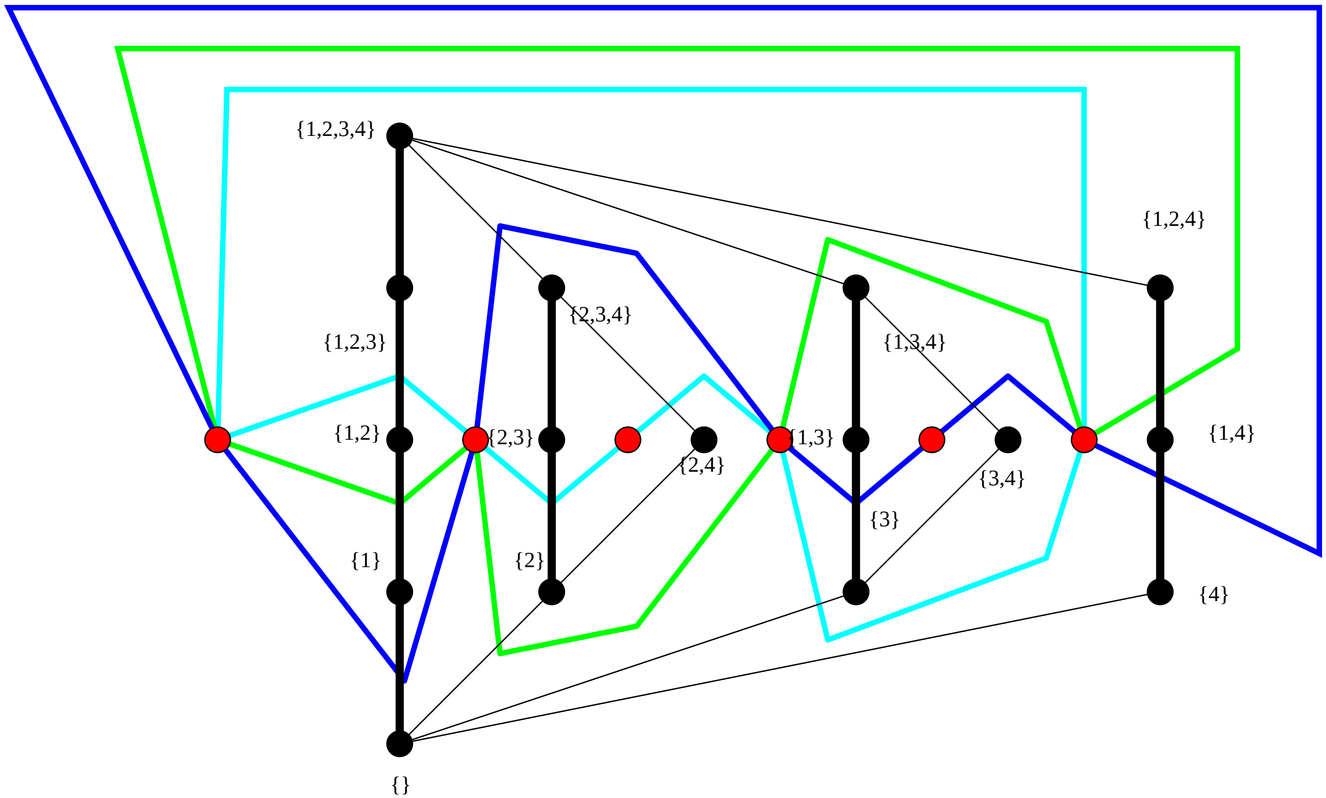
Take dual. First cross "1' edges.

Back to $\mathcal{B}_n$



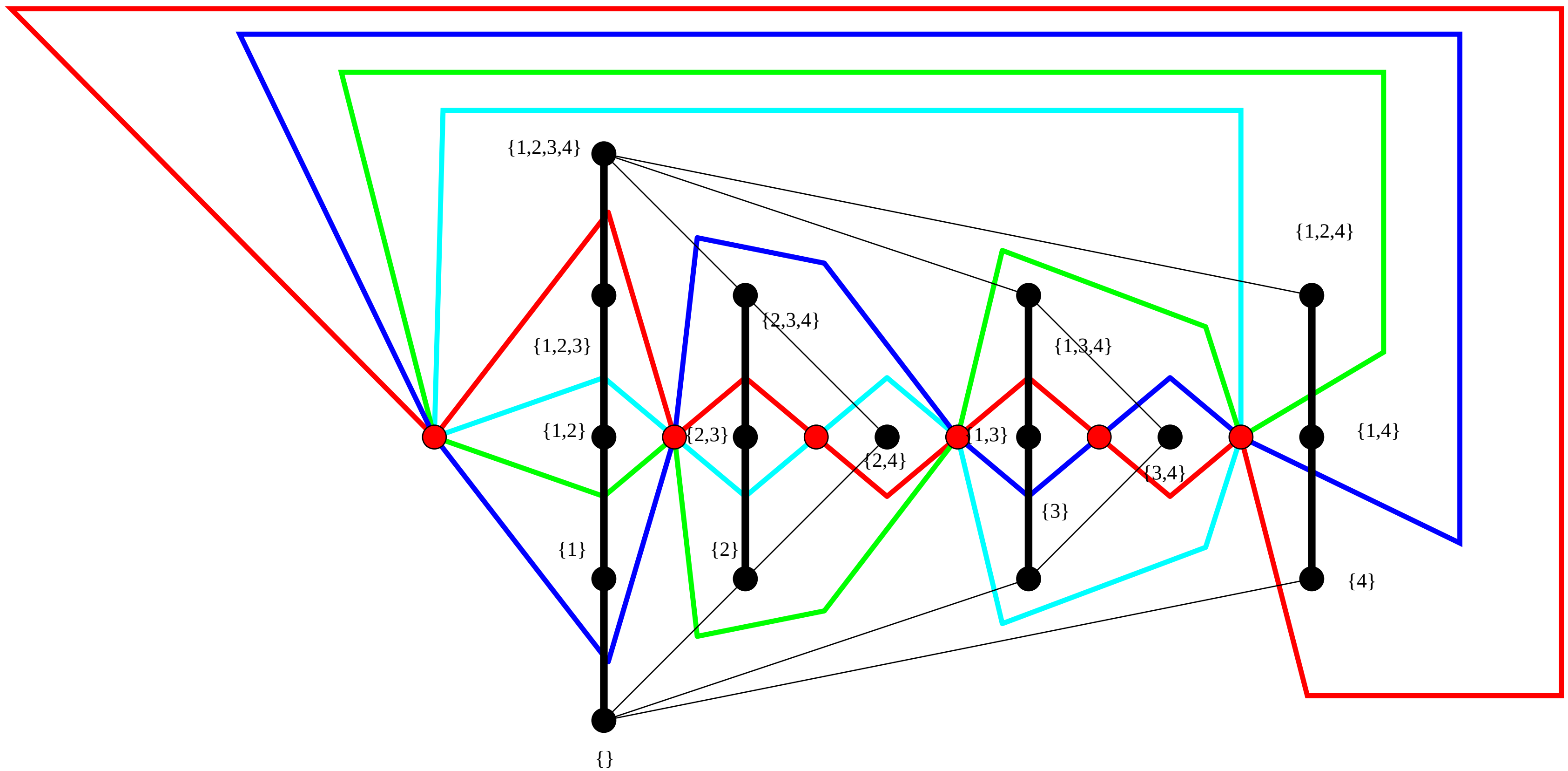
Take dual. First cross “1’ edges. Then cross “2 edges”.

Back to $\mathcal{B}_n$



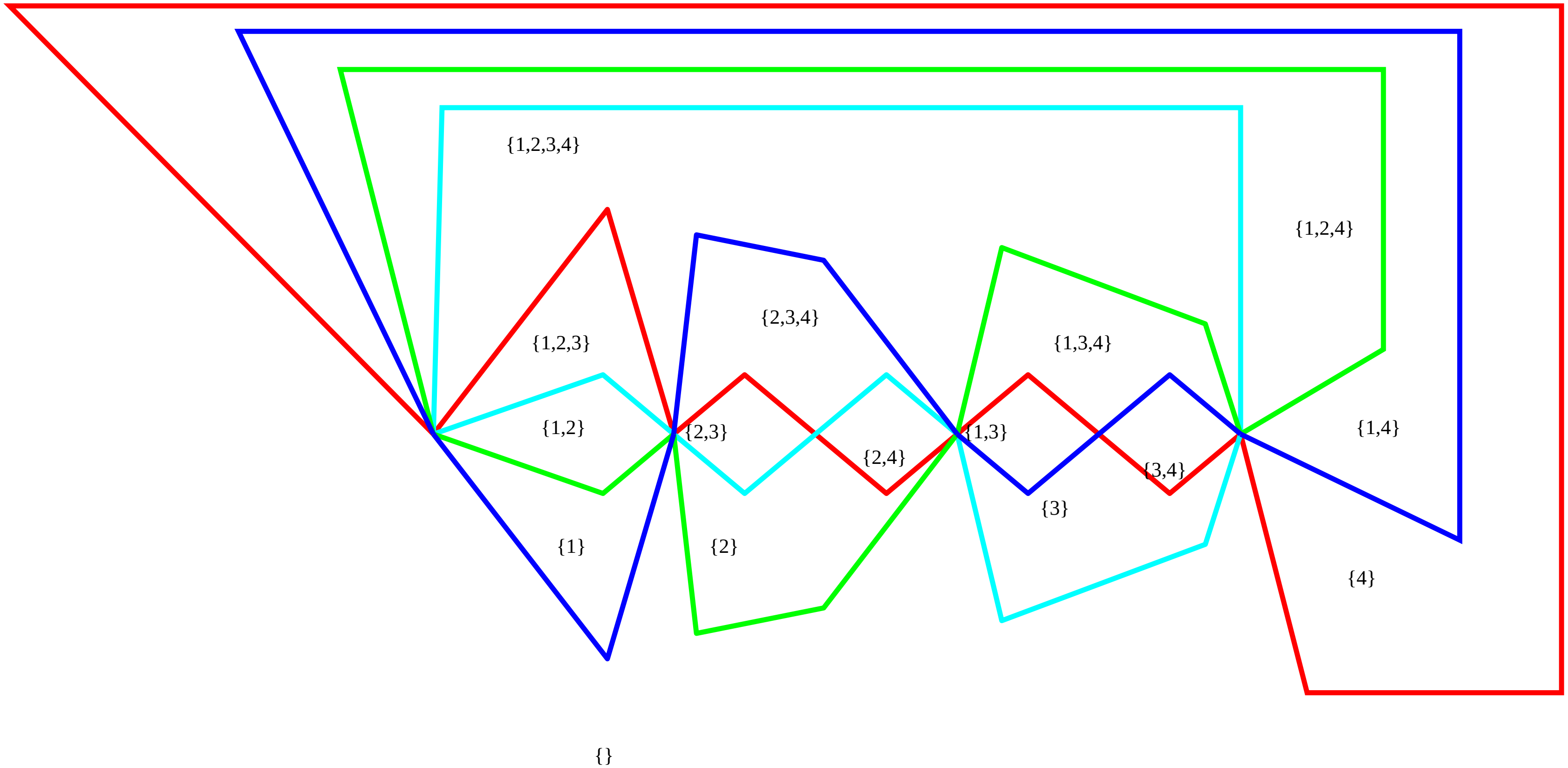
Take dual. First cross “1’ edges. Then cross “2 edges”. Etc.

Back to $\mathcal{B}_n$



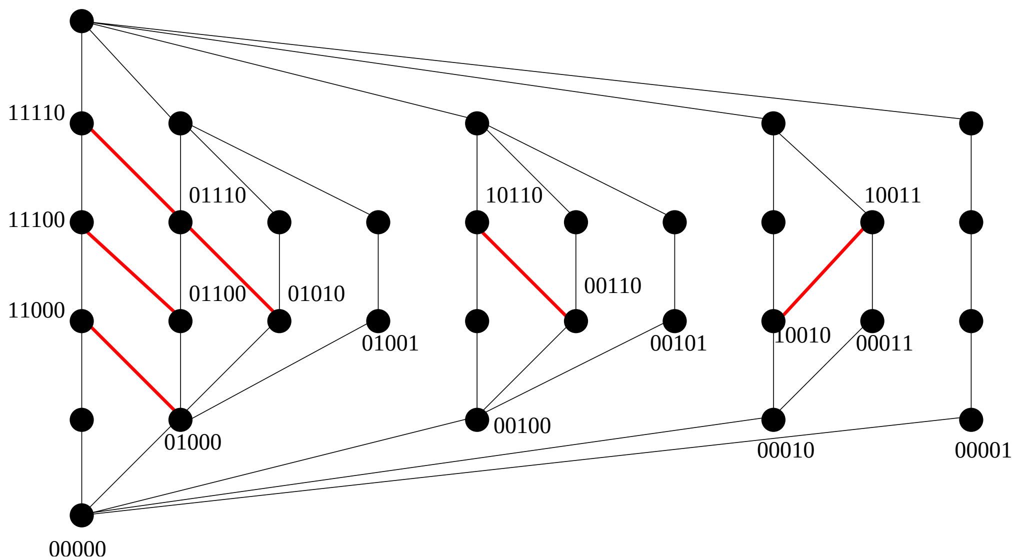
Take dual. First cross "1' edges. Then cross "2 edges". Etc.

Back to $\mathcal{B}_n$



Result: Venn diagram [GKS2004] (Can always find chain cover edges and planar embedding of the chain cover graph.)

Bonus



Theorem [KRSW 2004] The face bounded by a chain and its parent can always be "quadrangulated" by chords joining vertices which differ in one bit, giving a Venn diagram with $(2^n - 2)/2$ vertices (half simple).

Rotationally symmetric Venn diagrams?

No if n is composite [Henderson 1963].

For prime n?

Work in $1/n$ th of $\mathcal{B}_n$ (necklaces) to get SCD.

Embed in $1/n$ th pie slice of plane.

Rotate.

Necklace - equivalence class of bitstrings under rotation

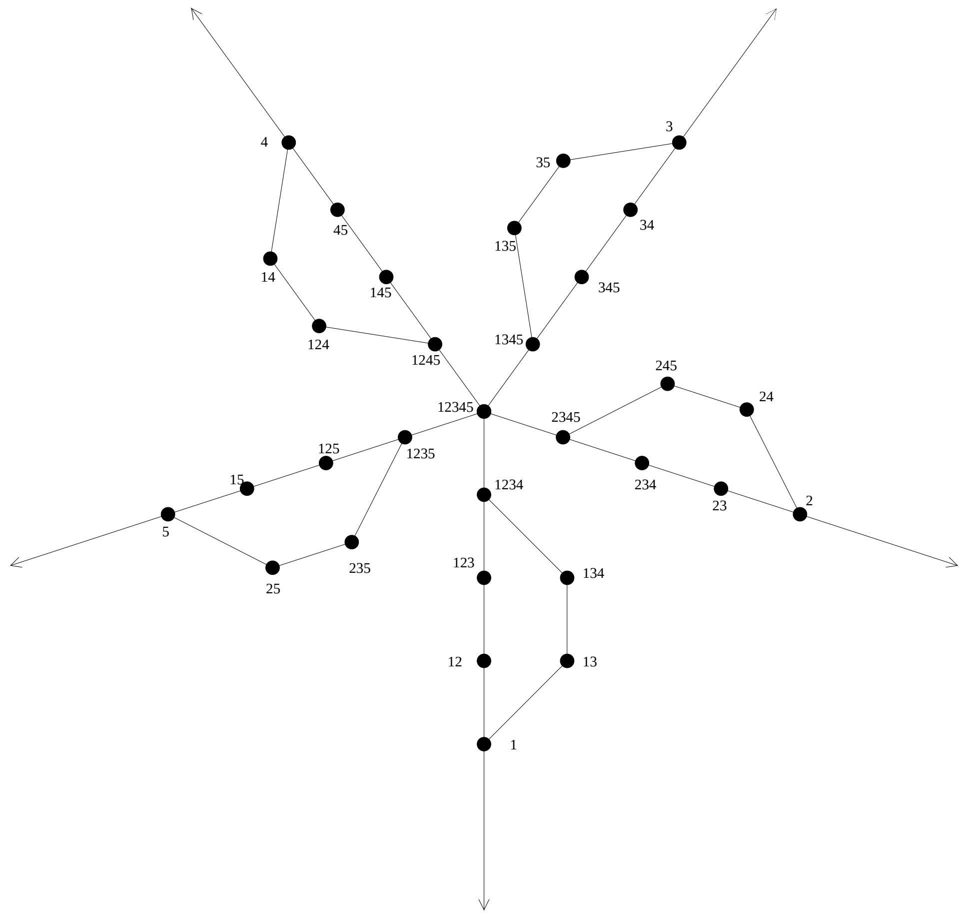
$\{11110, 01111, 10111, 11011, 11101\}$,

$\{10110, 01011, 10101, 11010, 01101\}$, etc.

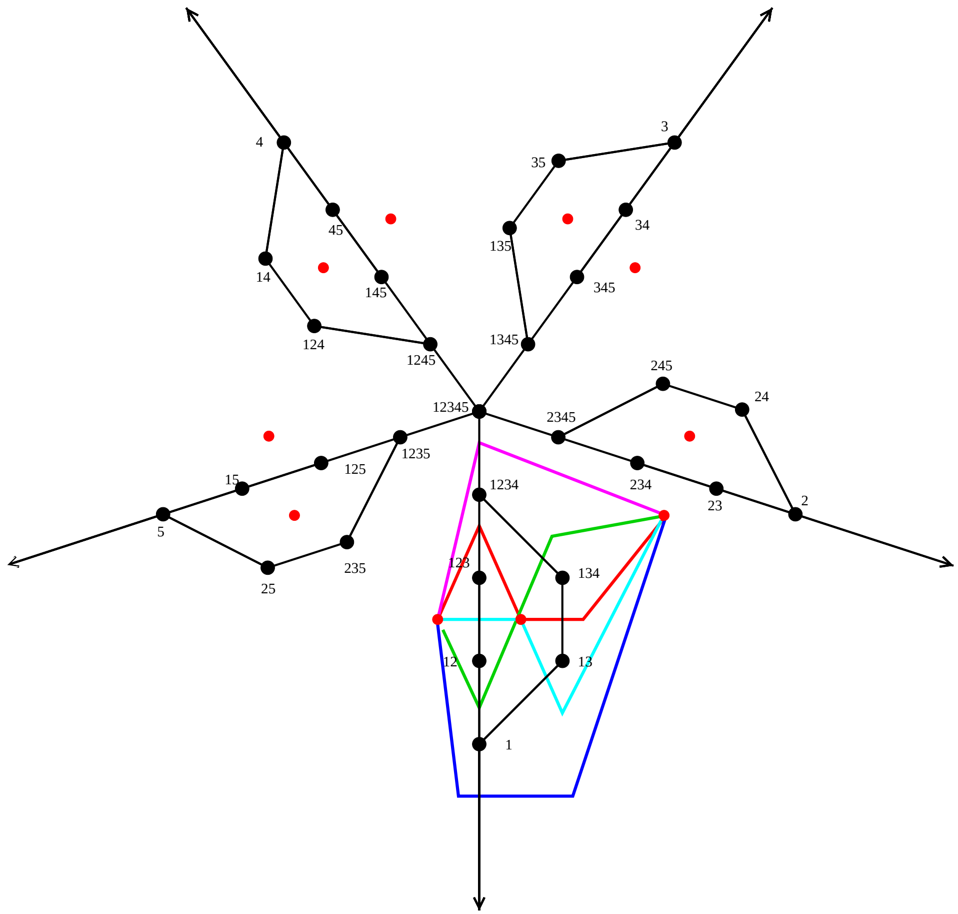
When n is prime:

if SCD for necklace reps.	then rotate once,	then three more times ...
11111		
11110	01111	
11100 10110	01110 01011	
11000 10100	01100 01010	
10000	01000	
00000		

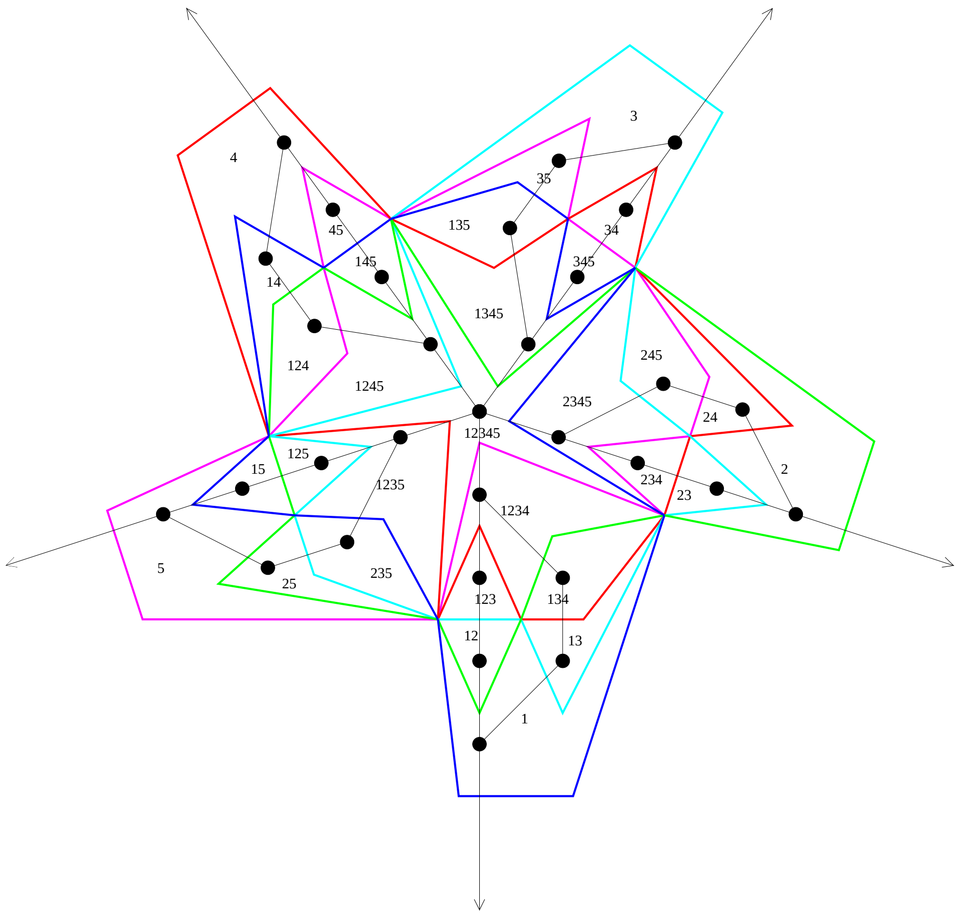
Getting symmetry from necklace SCD



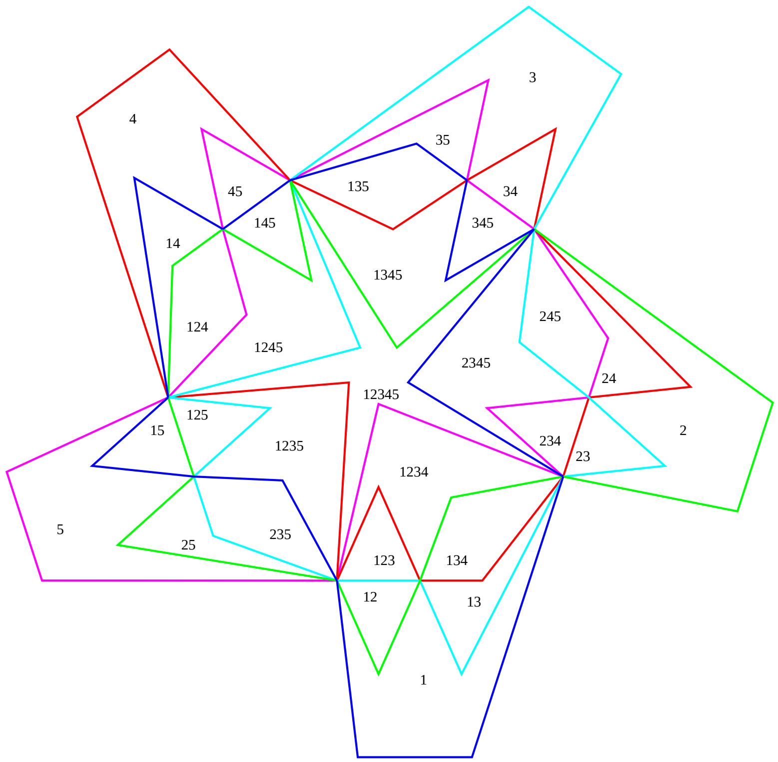
Getting symmetry from necklace SCD



Getting symmetry from necklace SCD



Getting symmetry from necklace SCD



Focus

How to choose necklace representatives so that the "necklace subposet" has a SCD with the chain cover property?

Solution

Choose as reps: bit string with the lex min *block code*. Then Greene-Kleitman works and you get:

Theorem [GKS 2004] Rotationally symmetric Venn diagrams exist for all prime n.

Block code of a binary string:

11000 1110 100000 10 110 $(5, 4, 6, 2, 3)$

0110001110100000101 1 (∞)

1011000111010000010 1 (∞)

110 11000 1110 100000 10 $(3, 5, 4, 6, 2)$

When n is prime:

- $x \in B_n$ has n distinct rotations

- no 2 rotations have same finite block code.

SCD in necklace poset for $n = 7$

		1111110		
	1011110	1111100	1101110	
1010110	1011100	1111000	1101100	1001110
1010100	1011000	1110000	1101000	1001100
	1010000	1100000	1001000	
		1000000		
(2,5)	(2,5)	(7)	(3,4)	(3,4)

G-K chains preserve paren matching **and** block code!

Necklace poset when n is composite - Does it still have a SCD?

	[111110] (6)		
[101110] (6)	[111100] (6)	[110110] (3)	
[101100] (6)	[111000] (6)	[110100] (6)	[101010] (2)
[101000] (6)	[110000] (6)	[100100] (3)	
	[100000] (6)		
(2,4)	(6)	(3,3)	(2,2,2)

(It shouldn't, but it seems to?)

For composite n,

Necklace poset is unimodal and symmetric [Stanley 1984].

[Jiang 2003]:

So are subposets induced by any block code.

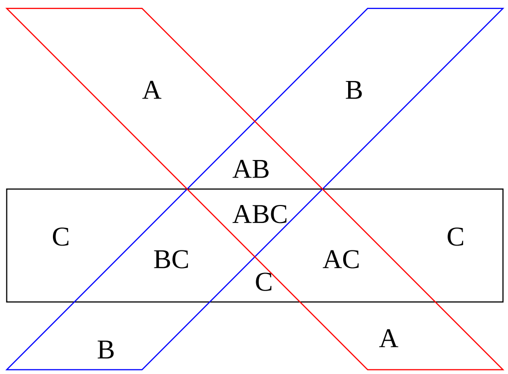
Block code aperiodic implies subposet has SCD.

Thus, suffices to check periodic block codes.

Necklace poset has SCD for all $n \leq 16$.

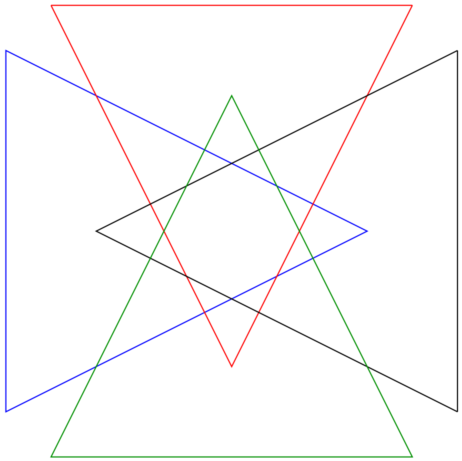
e.g. checking $n = 12$ ...

Independent family of curves (regions need not be connected)



Not a Venn diagram

(independent family of curves)



Symmetric independent family of curves

(but not a Venn diagram)

[Grunbaum 1999]:

The minimum number of regions in a rotationally symmetric independent family of curves is

$$2 + n(C_n - 2),$$

where C_n is the number of n-bit necklaces.

Rotationally symmetric independent families of curves can be constructed for any n.

But can this be done with the minimum number of regions?

Existence of a SCD with the chain cover property in the necklace poset would solve this problem.