Domination and Packing in Graph Products

Doug Rall

Department of Mathematics

Furman University

math.furman.edu/$\sim$drall

- Graph products
- Invariants
- Questions of Interest
- Domination
  - Cartesian products
  - Direct products
- Total domination
  - Cartesian products
  - Direct products
- Paired-domination
  - Cartesian products
- Open Problems

Cartesian Product

Given graphs G and H, the Cartesian product $G \square H$ has vertex set

$$V(G) \times V(H).$$

(g_1, h_1) is adjacent to (g_2, h_2) in $G \square H$ if

- $g_1 g_2$ is an edge of G and $h_1 = h_2$; or
- $g_1 = g_2$ and $h_1 h_2$ is an edge of H.

For $h \in V(H)$, the fiber G^h is the subgraph of $G \Box H$ induced by

$$\{(g,h) \mid g \in V(G)\}.$$

G^h is isomorphic to G.

The fiber ${}^gH$ is the subgraph of $G \Box H$ induced by

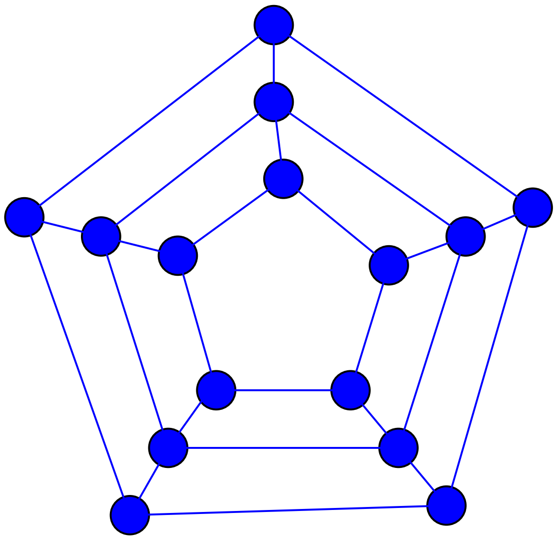
$$\{(g,h) \mid h \in V(H)\}.$$

${}^gH$ is isomorphic to H.

Product Graphs - Structure and Recognition (Imrich & Klavžar, 2000)

Cartesian: $G \square H$

$C_5 \square P_3$



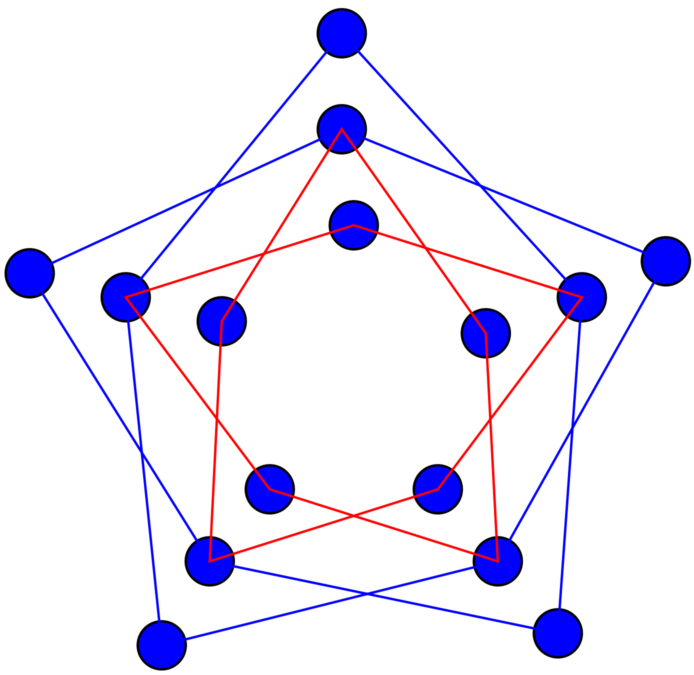
Direct Product: $G \times H$

$V(G \times H) = V(G) \times V(H)$

(g_1, h_1) is adjacent to (g_2, h_2) in $G \times H$ if

- $g_1 g_2$ is an edge of G and $h_1 h_2$ is an edge of H.

$C_5 \times P_3$



Domination Invariants

A set $D \subseteq V(G)$ is a dominating set if for every $x \in V(G) - D$ there exists a $y \in D$ adjacent to x. The domination number is the minimum cardinality, $\gamma(G)$, of a dominating set in G.

Domination Invariants

A set $D \subseteq V(G)$ is a dominating set if for every $x \in V(G) - D$ there exists a $y \in D$ adjacent to x. The domination number is the minimum cardinality, $\gamma(G)$, of a dominating set in G.

A set $S \subseteq V(G)$ is a total dominating set if for every $x \in V(G)$ there exists a $y \in S$ adjacent to x. The total domination number is the minimum cardinality, $\gamma_t(G)$, of a total dominating set in G.

Domination Invariants

A set $D \subseteq V(G)$ is a dominating set if for every $x \in V(G) - D$ there exists a $y \in D$ adjacent to x. The domination number is the minimum cardinality, $\gamma(G)$, of a dominating set in G.

A set $S \subseteq V(G)$ is a total dominating set if for every $x \in V(G)$ there exists a $y \in S$ adjacent to x. The total domination number is the minimum cardinality, $\gamma_t(G)$, of a total dominating set in G.

A set $P \subseteq V(G)$ is a paired dominating set if P is a dominating set and the subgraph of G induced by P contains a perfect matching. The paired domination number is the minimum cardinality, $\gamma_{\text{pr}}(G)$, of a paired dominating set in G.

A graphical invariant σ is supermultiplicative on a graph product $\oplus$ if for every pair of graphs G and H, $\sigma(G)\sigma(H) \leq \sigma(G \oplus H)$.

A graphical invariant σ is supermultiplicative on a graph product $\oplus$ if for every pair of graphs G and H, $\sigma(G)\sigma(H) \leq \sigma(G \oplus H)$.

A graphical invariant σ is submultiplicative on a graph product $\oplus$ if for every pair of graphs G and H, $\sigma(G \oplus H) \leq \sigma(G)\sigma(H)$.

I. Is the invariant σ supermultiplicative or submultiplicative on the product $\oplus$?

II. If σ is supermultiplicative on $\oplus$, does there exist a constant $K > 1$ such that

$$\sigma(G)\sigma(H) \leq \sigma(G \oplus H) \leq K \cdot \sigma(G)\sigma(H)?$$

III. If σ is submultiplicative on $\oplus$, does there exist a positive constant $k < 1$ such that

$$k \cdot \sigma(G)\sigma(H) \leq \sigma(G \oplus H) \leq \sigma(G)\sigma(H)?$$

IV. Is there a class of graphs $\mathcal{G}$ such that for every $G \in \mathcal{G}$

$$\sigma(G \oplus H) = \sigma(G)\sigma(H)$$

a. for some H ?

b. for all H ?

V. If all available evidence suggests that $\sigma(G)\sigma(H) \leq \sigma(G \oplus H)$ (but we cannot prove it), can we find a constant $0 < k < 1$ such that $k \cdot \sigma(G)\sigma(H) \leq \sigma(G \oplus H)$?

VI. If all available evidence suggests that $\sigma(G \oplus H) \leq \sigma(G)\sigma(H)$ (but we cannot prove it), can we find a constant $K > 1$ such that $\sigma(G \oplus H) \leq K \cdot \sigma(G)\sigma(H)$?

Domination in Cartesian Products

γ is **not** submultiplicative on $\square$.

$$1 = \gamma(K_3)\gamma(K_3) < 3 = \gamma(K_3 \square K_3)$$

It is not known if γ is supermultiplicative on $\square$, but all evidence suggests that it is.

Domination in Cartesian Products

γ is **not** submultiplicative on $\Box$.

$$1 = \gamma(K_3)\gamma(K_3) < 3 = \gamma(K_3 \Box K_3)$$

It is not known if γ is supermultiplicative on $\Box$, but all evidence suggests that it is.

In fact,

Vizing's Conjecture: $\gamma(G)\gamma(H) \leq \gamma(G \Box H)$

We say Vizing's conjecture holds for G if the above inequality holds for every H.

Conjecture outstanding since 1968.

Many authors have imposed conditions on G and shown that Vizing's conjecture holds for G.

Theorem: (Barcalkin and German, 1979) If G is a spanning subgraph of a graph K such that

- $\gamma(G) = \gamma(K)$ and
- $\gamma(K) = \chi(\overline{K})$,

then $\gamma(G)\gamma(H) \leq \gamma(G \square H)$ for every H.

Many authors have imposed conditions on G and shown that Vizing's conjecture holds for G.

Theorem: (Barcalkin and German, 1979) If G is a spanning subgraph of a graph K such that

- $\gamma(G) = \gamma(K)$ and
- $\gamma(K) = \chi(\overline{K})$, $\Longleftarrow$

then $\gamma(G)\gamma(H) \leq \gamma(G \square H)$ for every H.

Note that the second condition means that K has $\gamma(K)$ cliques that cover $V(K)$.

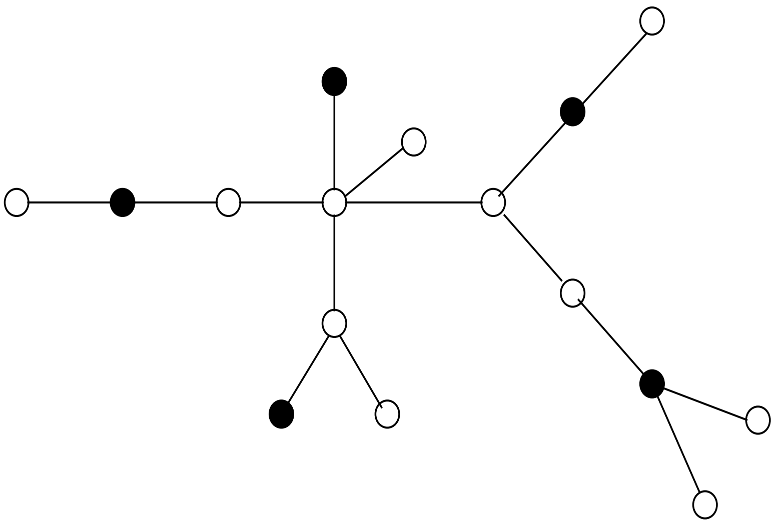
A set $A \subseteq V(G)$ is a 2-packing in G if for every $x \neq y$ in A, the closed neighborhoods $N[x]$ and $N[y]$ are disjoint. (Equivalently, $d(x, y) > 2$.) The 2-packing number is the maximum cardinality, $\rho_2(G)$, of a 2-packing in G.

In any graph G, $\rho_2(G) \leq \gamma(G)$ since any dominating set intersects every closed neighborhood.

Theorem: (Meir & Moon, 1975) For any tree T, $\rho_2(T) = \gamma(T)$.

Corollary: If T is a tree, then for every H

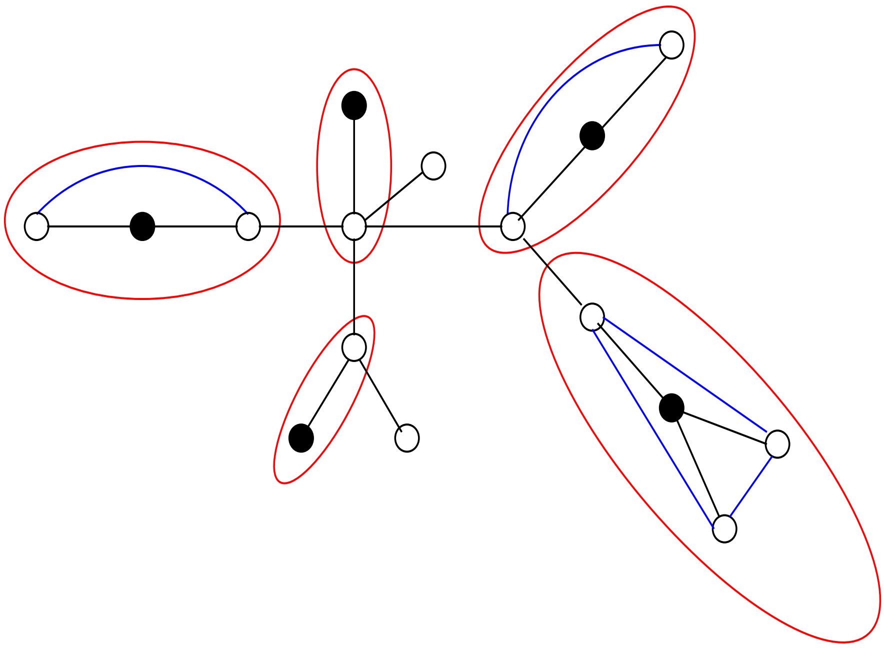
$$\gamma(T)\gamma(H) \leq \gamma(T \Box H)\,.$$



$$\gamma(T) = 5 = \rho_2(T)$$

Corollary: If T is a tree, then for every H

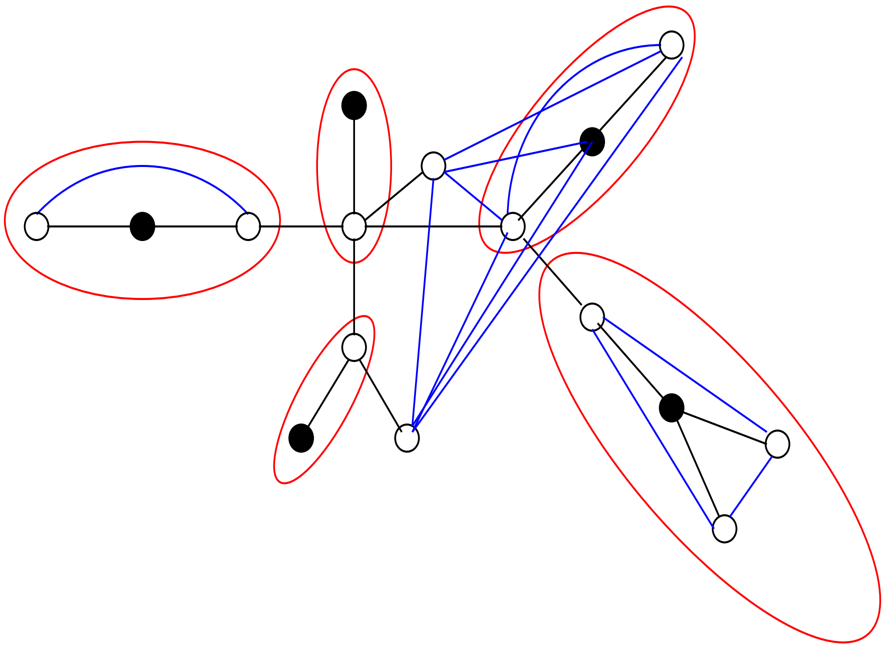
$$\gamma(T)\gamma(H) \leq \gamma(T \Box H)\,.$$



$$\gamma(T) = 5 = \rho_2(T)$$

Corollary: If T is a tree, then for every H

$$\gamma(T)\gamma(H) \leq \gamma(T \square H) .$$



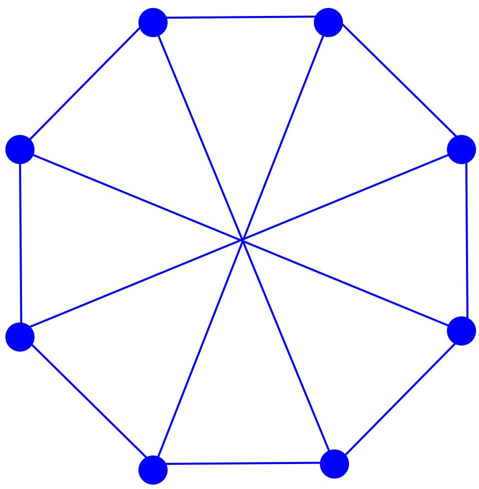
$$\gamma(T) = 5 = \rho_2(T)$$

Other conditions on G that imply that Vizing's conjecture holds for G:

- (B. & G., 1979) G is a cycle.
- (B. & G., 1979; Jacobson & Kinch, 1986) $\rho_2(G) = \gamma(G)$
- (Hartnell & D.R., 1995) $\rho_2(G) + 1 = \gamma(G)$
- (Liang Sun, 2004) $\gamma(G) = 3$
- (unpublished) $\chi(\overline{G}) \leq 4$
- (Ron Aharoni & Tibor Szabo, 2005) G is chordal

Note: It is not all that easy to find graphs G such that

- Vizing's conjecture holds for G, and
- the condition from the theorem of Barcalkin & German does not hold for G.



V. Is there a constant $0 < k < 1$ such that

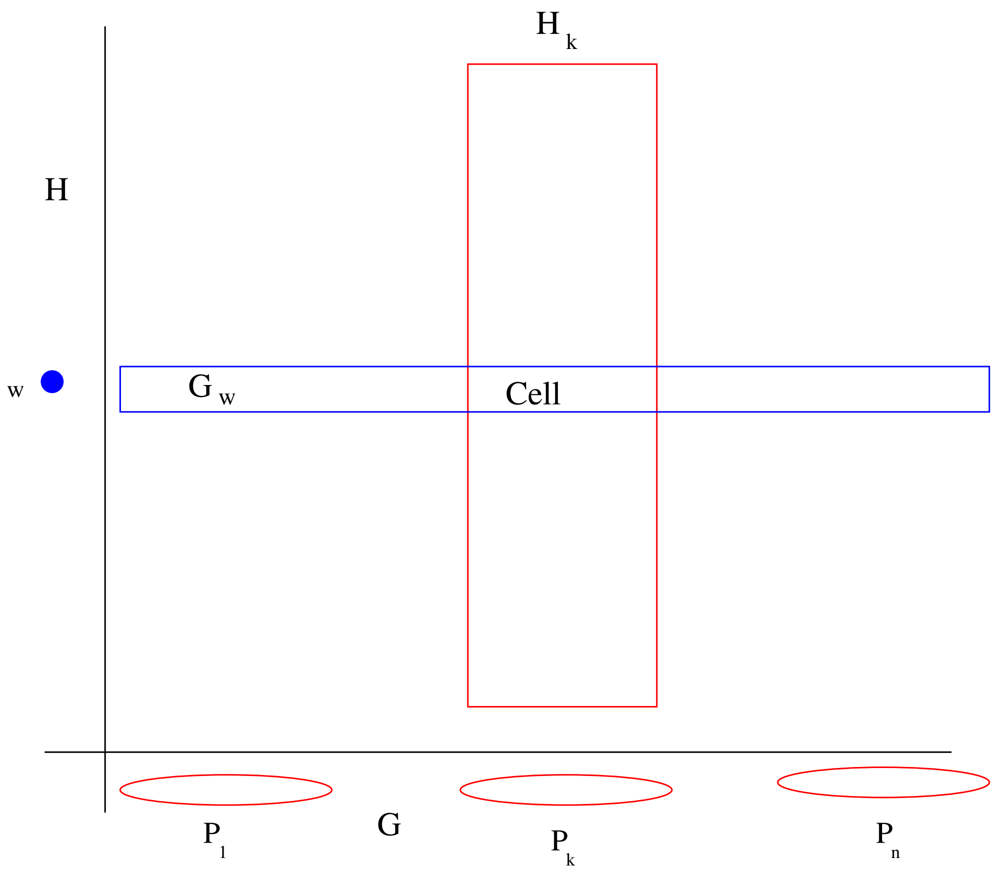
$$k \cdot \gamma(G)\gamma(H) \leq \gamma(G \Box H)?$$

Theorem: (Clark & Suen, 2000)
$\frac{1}{2}\gamma(G)\gamma(H) \leq \gamma(G \Box H)$ for every G and H.

V. Is there a constant $0 < k < 1$ such that

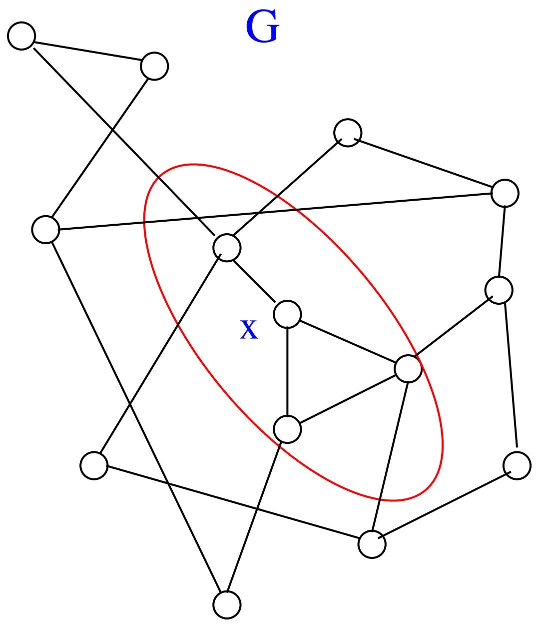
$$k \cdot \gamma(G)\gamma(H) \leq \gamma(G \Box H)?$$

Theorem: (Clark & Suen, 2000)
$\frac{1}{2}\gamma(G)\gamma(H) \leq \gamma(G \Box H)$ for every G and H.



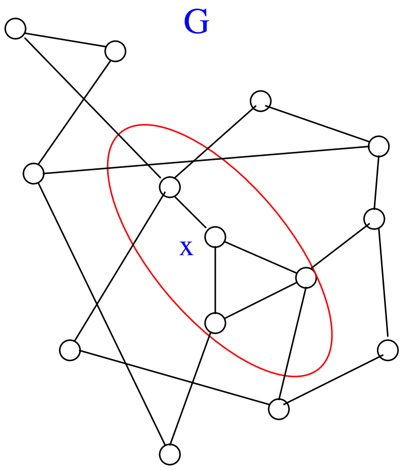
Theorem: (Jacobson & Kinch, 1986)

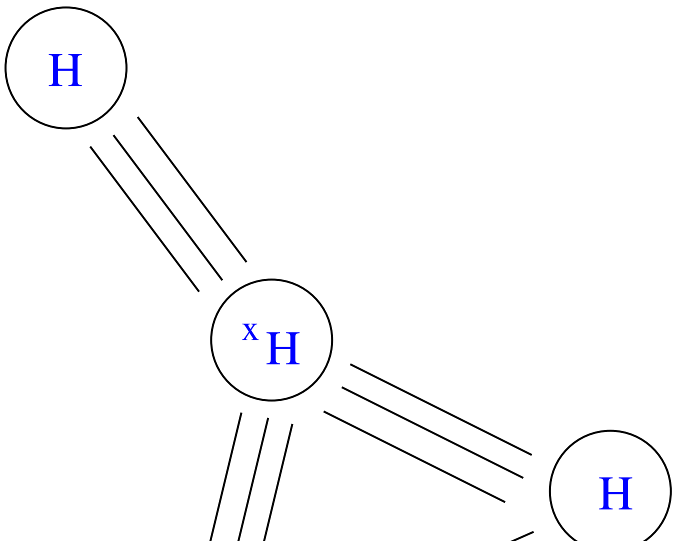
$\gamma(H)\rho_2(G) \leq \gamma(G \square H)$



Theorem: (Jacobson & Kinch, 1986)

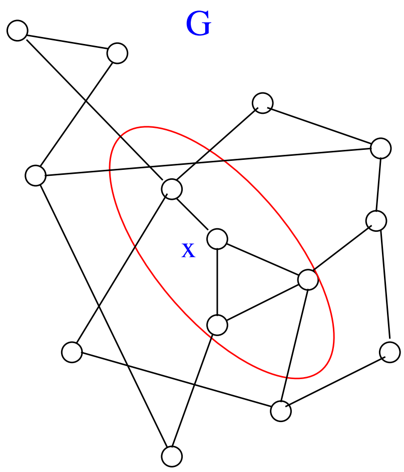
$\gamma(H)\rho_2(G) \leq \gamma(G \square H)$

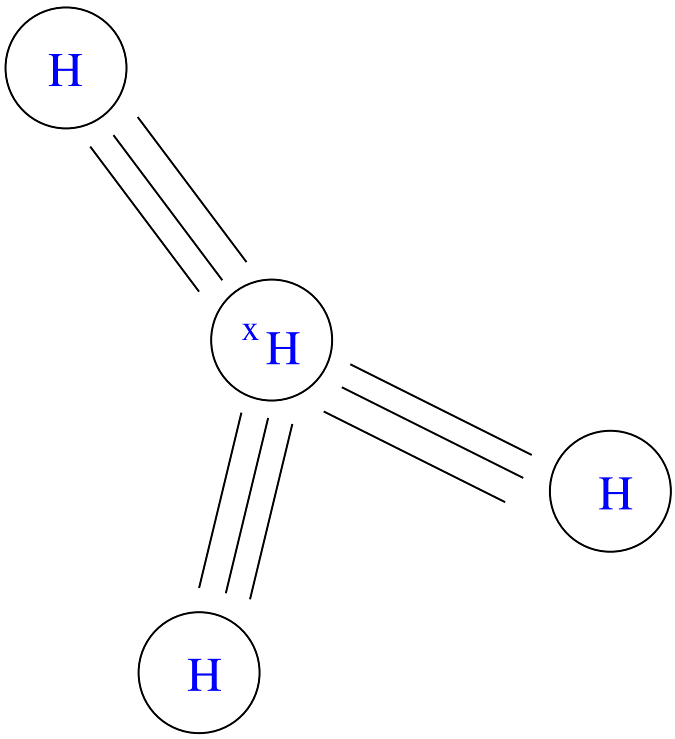




Theorem: (Jacobson & Kinch, 1986)

$\gamma(H)\rho_2(G) \leq \gamma(G \square H)$





Corollary: If $\rho_2(G) = \gamma(G)$, then for all H

$$\gamma(H)\gamma(G) \leq \gamma(G \square H)\,.$$

In particular, for a tree T and any H,

$$\gamma(H)\gamma(T) \leq \gamma(T \square H)\,.$$

Corollary: If $\rho_2(G) = \gamma(G)$, then for all H

$$\gamma(H)\gamma(G) \leq \gamma(G \Box H)\,.$$

In particular, for a tree T and any H,

$$\gamma(H)\gamma(T) \leq \gamma(T \Box H)\,.$$

There is no constant $K > 1$ such that for all G, H

$$\gamma(G \Box H) < K \cdot \gamma(G)\gamma(H)\,.$$

Example:

$$\gamma(K_n \Box K_n) = n \cdot \gamma(K_n)\gamma(K_n)\,.$$

Summary

2-Packing & Domination on $\Box$

- $\rho_2(G) \leq \gamma(G)$ for every graph G.

- $\rho_2(G)\gamma(H) \leq \gamma(G \Box H)$ for every H.

- $\rho_2(T) = \gamma(T)$ for every tree T.

Domination in Direct Products

γ is **not** submultiplicative on $\times$.

$$\gamma(K_3 \times K_3) = 3 > \gamma(K_3)\gamma(K_3)$$

γ is **not** supermultiplicative on $\times$.

Let $G = K_6 - 3K_2$. Then

$$\gamma(G \times G) = 3 < \gamma(G)\gamma(G)$$

Domination in Direct Products

γ is **not** submultiplicative on $\times$.

$$\gamma(K_3 \times K_3) = 3 > \gamma(K_3)\gamma(K_3)$$

γ is **not** supermultiplicative on $\times$.

Let $G = K_6 - 3K_2$. Then

$$\gamma(G \times G) = 3 < \gamma(G)\gamma(G)$$

(Nowakowski & D.R., 1996) If $\gamma(G) = \rho_2(G)$, then for all H,

$$\gamma(G \times H) \geq \gamma(G)\gamma(H)\,.$$

From

$\gamma_t(G) \le 2\gamma(G)$ (immediate) and

γ_t is submultiplicative on $\times$ (later)

follows

Observation:

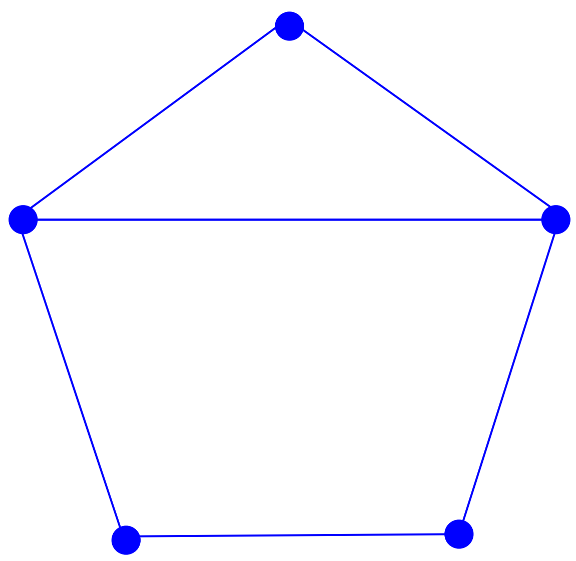
$$\gamma(G \times H) \le \gamma_t(G \times H) \le \gamma_t(G)\gamma_t(H) \le 4\,\gamma(G)\gamma(H)$$

Is this bound sharp?

Theorem: (Brešar, Klavžar & D.R., 2004)
For all graphs G and H,

$$\gamma(G \times H) \leq 3\,\gamma(G)\gamma(H)\,,$$

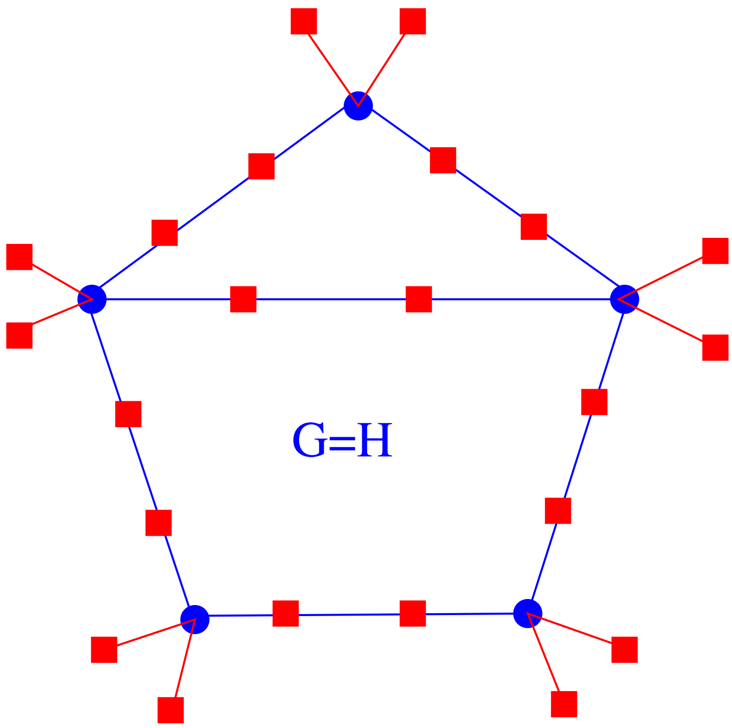
and this bound is sharp.



Theorem: (Brešar, Klavžar & D.R., 2004)
For all graphs G and H,

$$\gamma(G \times H) \leq 3\,\gamma(G)\gamma(H)\,,$$

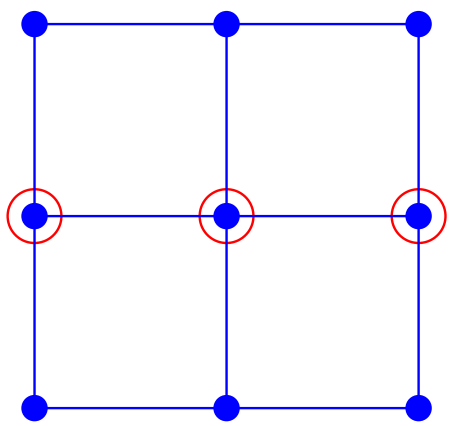
and this bound is sharp.



Total Domination in Cartesian Products

γ_t is **not** supermultiplicative on $\Box$.

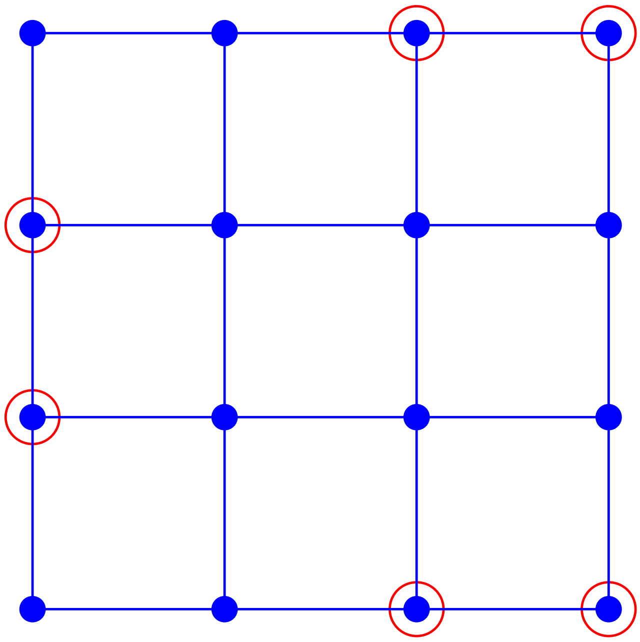
$$\gamma_t(P_3 \Box P_3) = 3 < 2 \cdot 2 = \gamma_t(P_3)\gamma_t(P_3)$$



Total Domination in Cartesian Products

γ_t is **not** submultiplicative on $\Box$.

$$\gamma_t(P_4 \Box P_4) = 6 > 2 \cdot 2 = \gamma_t(P_4)\gamma_t(P_4)$$



6 pairwise disjoint open neighborhoods

Proposition

$$\begin{aligned} \gamma_t(G)\gamma_t(H) &\le 4\,\gamma(G)\gamma(H) \\ &\le 8\,\gamma(G\Box H) \quad \textit{Clark, Suen} \\ &\le 8\,\gamma_t(G\Box H) \end{aligned}$$

Proposition

$$\begin{aligned}\gamma_t(G)\gamma_t(H) &\leq 4\,\gamma(G)\gamma(H)\\ &\leq 8\,\gamma(G\Box H) \quad \textit{Clark, Suen}\\ &\leq 8\,\gamma_t(G\Box H)\end{aligned}$$

OR

$$\frac{1}{8}\gamma_t(G)\gamma_t(H) \leq \gamma_t(G\Box H)$$

Find the largest constant k such that for all G and H with no isolated vertices

$$k \cdot \gamma_t(G)\gamma_t(H) \leq \gamma_t(G \Box H).$$

- $k \geq \frac{1}{8}$ by previous observation.
- $k \leq \frac{3}{4}$ since $\frac{3}{4}\gamma_t(P_3)\gamma_t(P_3) = \gamma_t(P_3 \Box P_3)$

Find the largest constant k such that for all G and H with no isolated vertices

$$k \cdot \gamma_t(G)\gamma_t(H) \leq \gamma_t(G \square H).$$

- $k \geq \frac{1}{8}$ by previous observation.
- $k \leq \frac{3}{4}$ since $\frac{3}{4}\gamma_t(P_3)\gamma_t(P_3) = \gamma_t(P_3 \square P_3)$

Theorem: (Henning, D.R., 2003) For any graphs G and H without isolated vertices,

$$\frac{1}{6}\gamma_t(G)\gamma_t(H) \leq \gamma_t(G \square H).$$

Theorem: (M.H., D.R., 2003) If $\rho_2(G) = \gamma(G)$ and H has no isolated vertices, then

$$\frac{1}{2}\gamma_t(G)\gamma_t(H) \leq \gamma_t(G \Box H).$$

Corollary: (M.H., D.R., 2003) If T is a nontrivial tree and H has no isolated vertices, then

$$\frac{1}{2}\gamma_t(T)\gamma_t(H) \leq \gamma_t(T \Box H).$$

Equality occurs $\Leftrightarrow$ $\gamma_t(T) = 2\gamma(T)$ and $H = n\,K_2$.

We know of **no** graphs G and H such that

$$\frac{1}{6}\gamma_t(G)\gamma_t(H) \leq \gamma_t(G \square H) < \frac{1}{2}\gamma_t(G)\gamma_t(H)$$

Is it the case that for all graphs with no isolated vertices

$$\frac{1}{2}\gamma_t(G)\gamma_t(H) \leq \gamma_t(G \square H)?$$

Total Domination in Direct Products

A is a total dominating set of G and B is a total dominating set of H

$\Rightarrow$

$A \times B$ is a total dominating set of $G \times H$

$\Rightarrow$

$\gamma_t(G \times H) \leq \gamma_t(G)\gamma_t(H)$

γ_t **is** submultiplicative on $\times$.

III. Does there exist a positive constant $k < 1$ such that

$$k \cdot \gamma_t(G)\gamma_t(H) \leq \gamma_t(G \times H)?$$

IV Is there a class of graphs $\mathcal{G}$ such that for every $G \in \mathcal{G}$

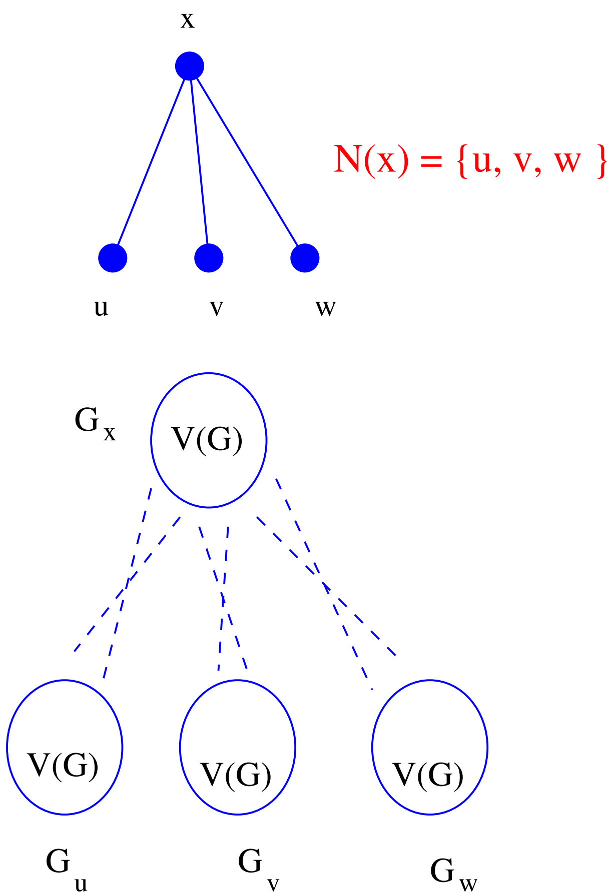
$$\gamma_t(G \times H) = \gamma_t(G)\gamma_t(H)$$

for all H?

Let D be a total dominating set of $G \times H$, and let $x \in V(H)$.

D totally dominates $G_x = \{(g, x) \mid g \in V(G)\}$ $\Rightarrow$

$$|D \cap (G_u \cup G_v \cup G_w)| \geq \gamma_t(G)$$

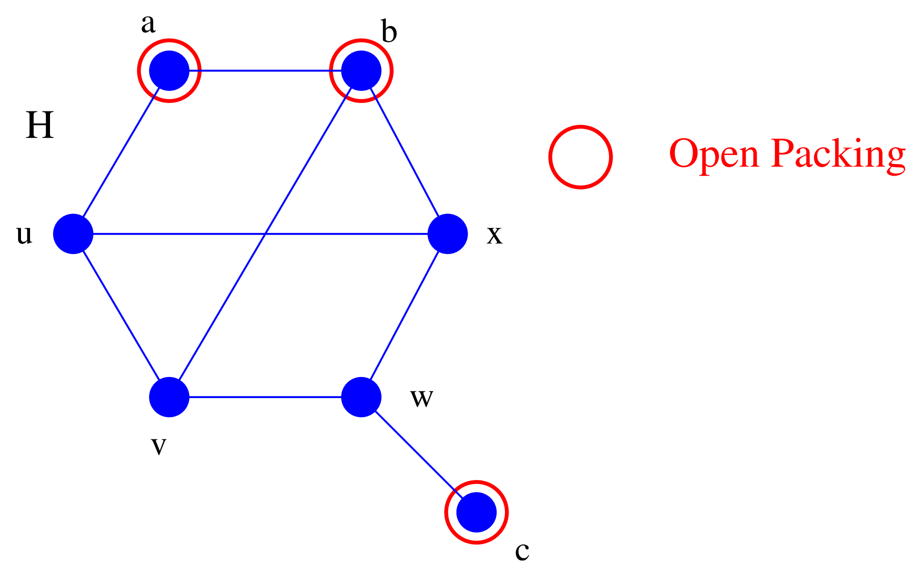


The open packing number of H is the maximum number, $\rho^{\circ}(H)$, of pairwise disjoint, open neighborhoods in H.

Lemma:
$\gamma_t(G \times H) \geq \max\{\rho^{\circ}(G)\gamma_t(H), \rho^{\circ}(H)\gamma_t(G)\}$

The open packing number of H is the maximum number, $\rho^{o}(H)$, of pairwise disjoint, open neighborhoods in H.

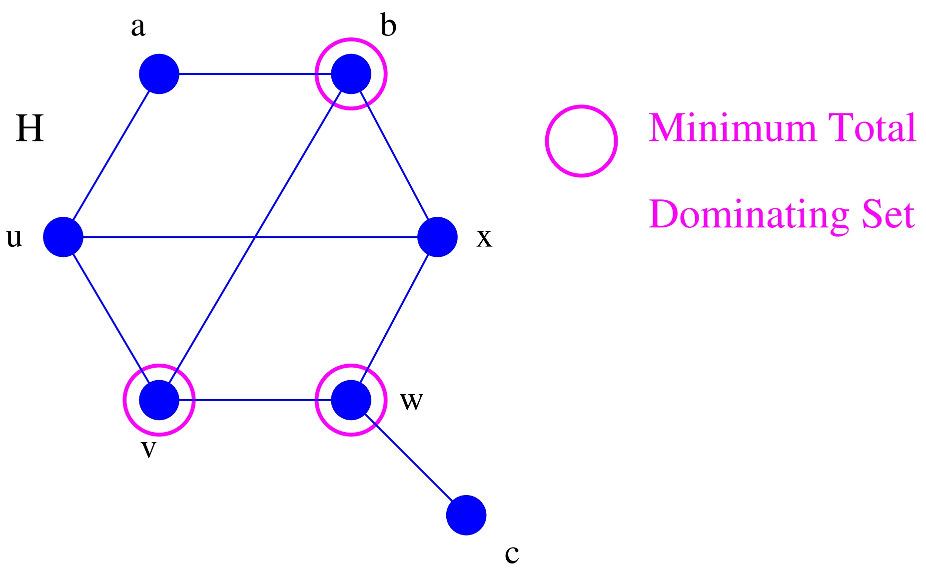
Lemma:
$\gamma_t(G \times H) \geq \max\{\rho^{o}(G)\gamma_t(H), \rho^{o}(H)\gamma_t(G)\}$



The open packing number of H is the maximum number, $\rho^{o}(H)$, of pairwise disjoint, open neighborhoods in H.

Lemma:
$\gamma_t(G \times H) \geq \max\{\rho^{o}(G)\gamma_t(H), \rho^{o}(H)\gamma_t(G)\}$

$3\gamma_t(G) \leq \max\{\rho^{o}(G)\gamma_t(H), 3\gamma_t(G)\} \leq \gamma_t(G \times H) \leq \gamma_t(G)\gamma_t(H) = 3\gamma_t(G)$



Observation: For every G, $\rho^{o}(G) \leq \gamma_t(G)$.

Theorem: (D.R., 2003) If $\rho^{o}(G) = \gamma_t(G)$, then for any H

$$\gamma_t(G \times H) = \gamma_t(G)\gamma_t(H)$$

Which graphs have $\rho^{o}(G) = \gamma_t(G)$?

Observation: For every G, $\rho^{\circ}(G) \leq \gamma_t(G)$.

Theorem: (D.R., 2003) If $\rho^{\circ}(G) = \gamma_t(G)$, then for any H

$$\gamma_t(G \times H) = \gamma_t(G)\gamma_t(H)$$

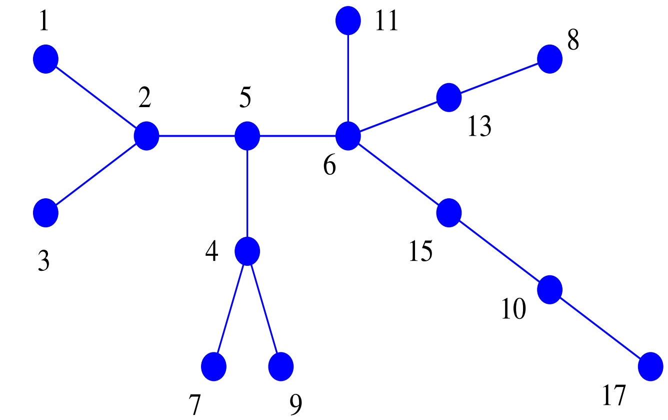
Which graphs have $\rho^{\circ}(G) = \gamma_t(G)$?

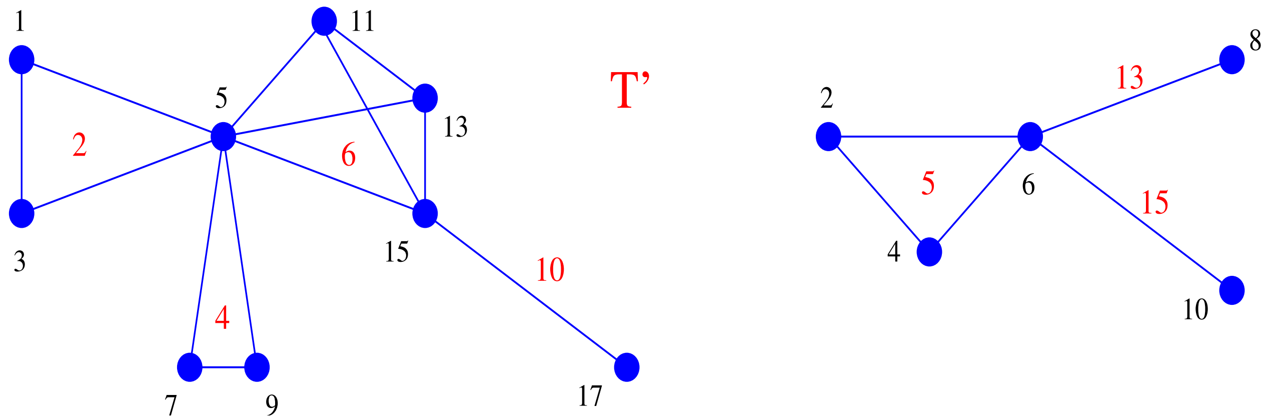
For $G = (V, E)$ let G' be the intersection graph on the collection of open neighborhoods of G.

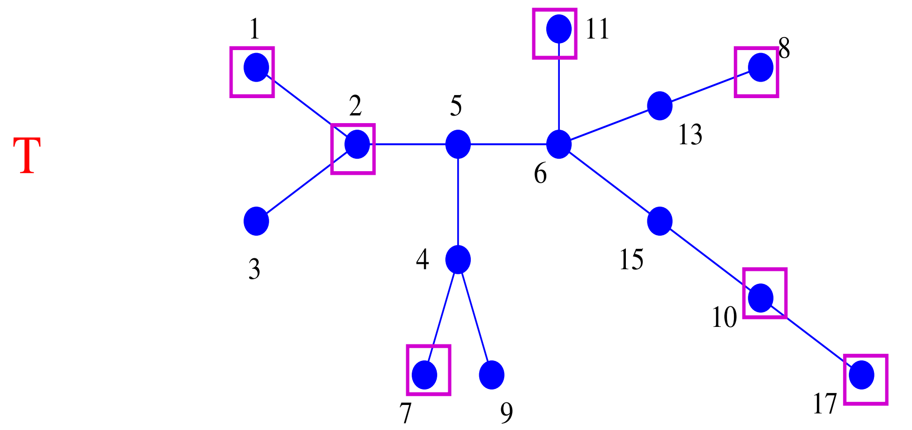
- $\rho^{\circ}(G) = \alpha(G')$ ($\alpha=$ independence number)

- $\gamma_t(G) =$ the minimum number of open neighborhoods that contain (cover) $V(G)$.

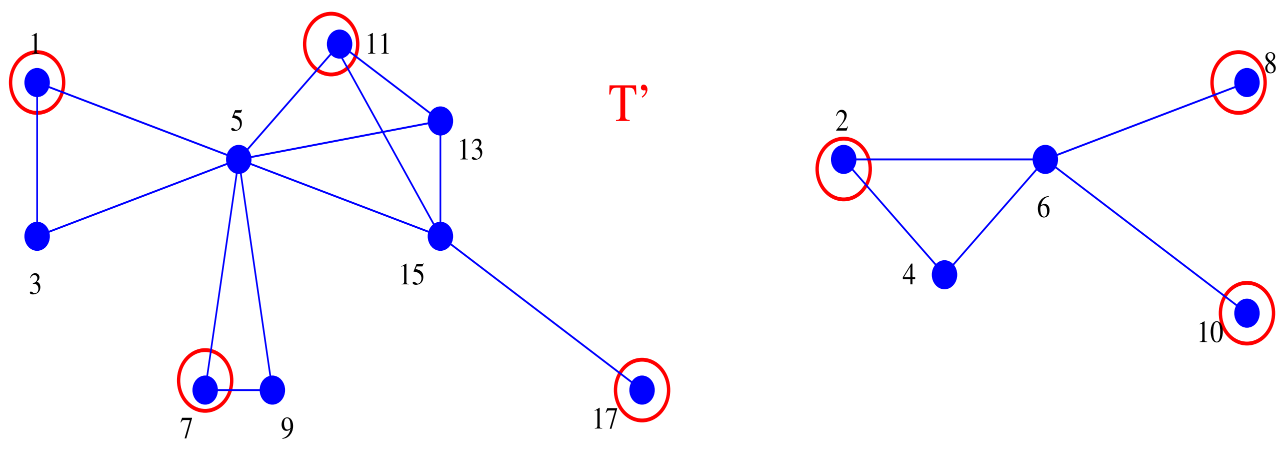
T



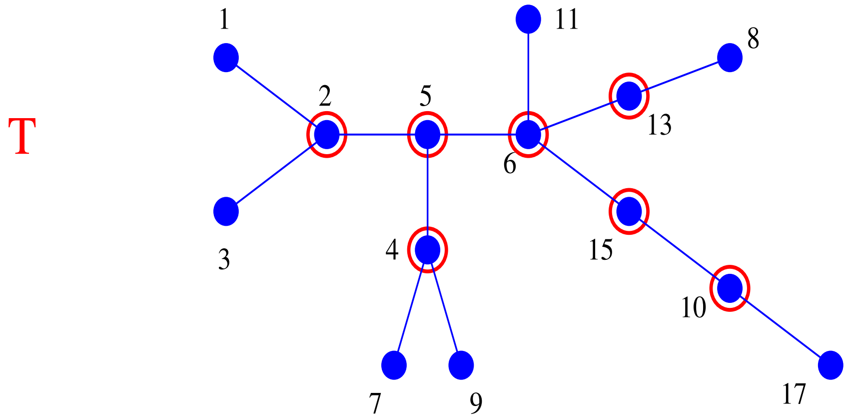


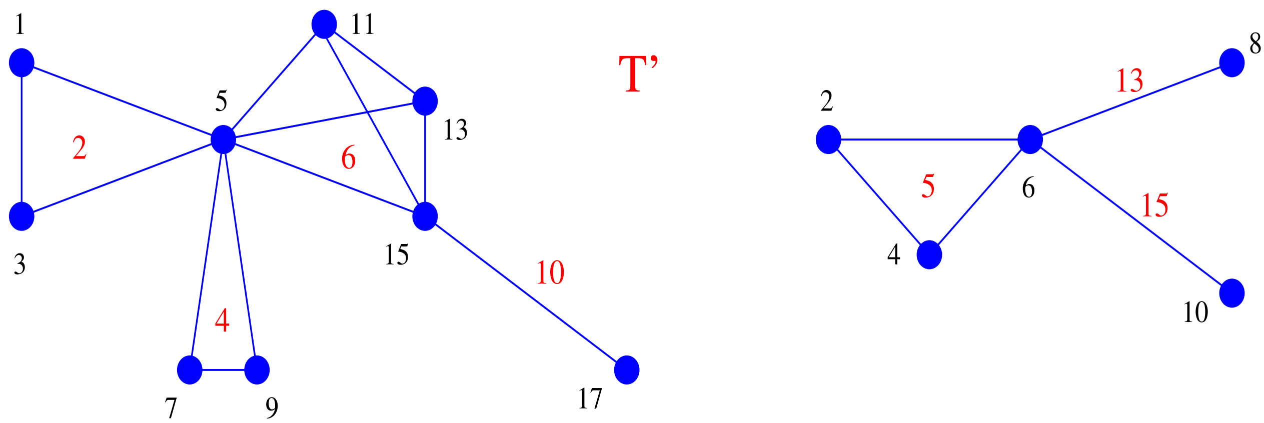


{ 1, 2, 7, 8, 10, 11, 17 } is maximum open packing in T



{1, 7, 11, 17, 2, 8, 10 } is maximum independent set in T'





{ 2, 4, 5, 6, 10, 13, 15 } is a total dominating set of T

Theorem: (D.R., 2003) For any nontrivial tree T, $\rho^{\circ}(T) = \gamma_t(T)$.

Theorem: (D.R., 2003) For any nontrivial tree T, $\rho^{\circ}(T) = \gamma_t(T)$.

Theorem: (Meir, Moon 1975) For any tree T,

$$\rho_2(T) = \gamma(T).$$

Their proof is by induction on the order of T.

Theorem: (D.R., 2003) For any nontrivial tree T, $\rho^{\circ}(T) = \gamma_t(T)$.

Proof (Sketch)

- For any clique M of T' there is a vertex a of T such that $M \subseteq N_T(a)$.
- $\gamma_t(T) =$ minimum number of cliques of T' that cover $V(T')$.
- T' is a chordal (and hence perfect) graph.
- $\overline{T'}$ is perfect.
- $\rho^{\circ}(T) = \alpha(T') = \omega(\overline{T'}) = \chi(\overline{T'})$
- $\chi(\overline{T'}) =$ mimimum number of independent sets that cover $\overline{T'} =$ minimum number of cliques that cover $V(T') = \gamma_t(T)$.

Corollary: If T is any nontrivial tree, then for any H

$$\gamma_t(T \times H) = \gamma_t(T)\gamma_t(H).$$

"direct grids"

$$\gamma_t(P_n \times P_m) = \gamma_t(P_n)\gamma_t(P_m)$$

(Example:)

$$\gamma_t(P_{17} \times P_{12}) = \gamma_t(P_{17})\gamma_t(P_{12}) = 9 \cdot 6 = 54$$

Corollary: For every H,

$$\gamma_t(C_{4n} \times H) = \gamma_t(C_{4n})\gamma_t(H) = 2n\gamma_t(H).$$

Summary
Open Packing & Total Domination on $\times$

- $\rho^{\circ}(G) \leq \gamma_t(G)$ for every graph G.

- $\rho^{\circ}(G)\gamma_t(H) \leq \gamma_t(G \times H)$ for every H.

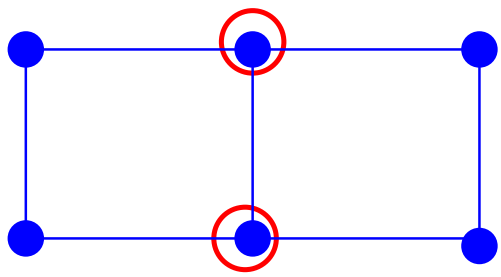
- $\rho^{\circ}(T) = \gamma_t(T)$ for every tree T.

Paired-Domination in Cartesian Products

Boštjan Brešar, Mike Henning, D.R., 2004

γ_{pr} is **not** supermultiplicative on $\Box$.

$$\gamma_{\text{pr}}(P_2 \Box P_3) = 2 < 2 \cdot 2 = \gamma_{\text{pr}}(P_2)\gamma_{\text{pr}}(P_3)$$



γ_{pr} is **not** submultiplicative on $\Box$.

$$\gamma_{\text{pr}}(K_n \Box K_n) \geq n > 2 \cdot 2 = \gamma_{\text{pr}}(K_n)\gamma_{\text{pr}}(K_n)$$

Proposition:(Haynes, Slater 1998)
$\gamma_{\mathrm{pr}}(G) \leq 2\gamma(G)$

Observation:

$$\begin{aligned}\gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(H) &\leq 4\,\gamma(G)\gamma(H)\\ &\leq 8\,\gamma(G\Box H) \quad \mathit{Clark, Suen}\\ &\leq 8\,\gamma_{\mathrm{pr}}(G\Box H)\end{aligned}$$

OR

$$\frac{1}{8}\gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(H) \leq \gamma_{\mathrm{pr}}(G\Box H)$$

What is largest constant $\frac{1}{8} \leq k < 1$ such that

$$k \cdot \gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(H) \leq \gamma_{\mathrm{pr}}(G\Box H)\,?$$

A set $A \subseteq V(G)$ is a 3-packing in G if for every $x \neq y$ in A, $d(x, y) > 3$. The 3-packing number is the maximum cardinality, $\rho_3(G)$, of a 3-packing in G.

Lemma: For any G with no isolated vertices,

$$\gamma_{\mathrm{pr}}(G) \geq 2\rho_3(G)\,.$$

Theorem: For any tree T,

$$\gamma_{\mathrm{pr}}(T) = 2\rho_3(T)\,.$$

Theorem: For any G and H with no isolated vertices,

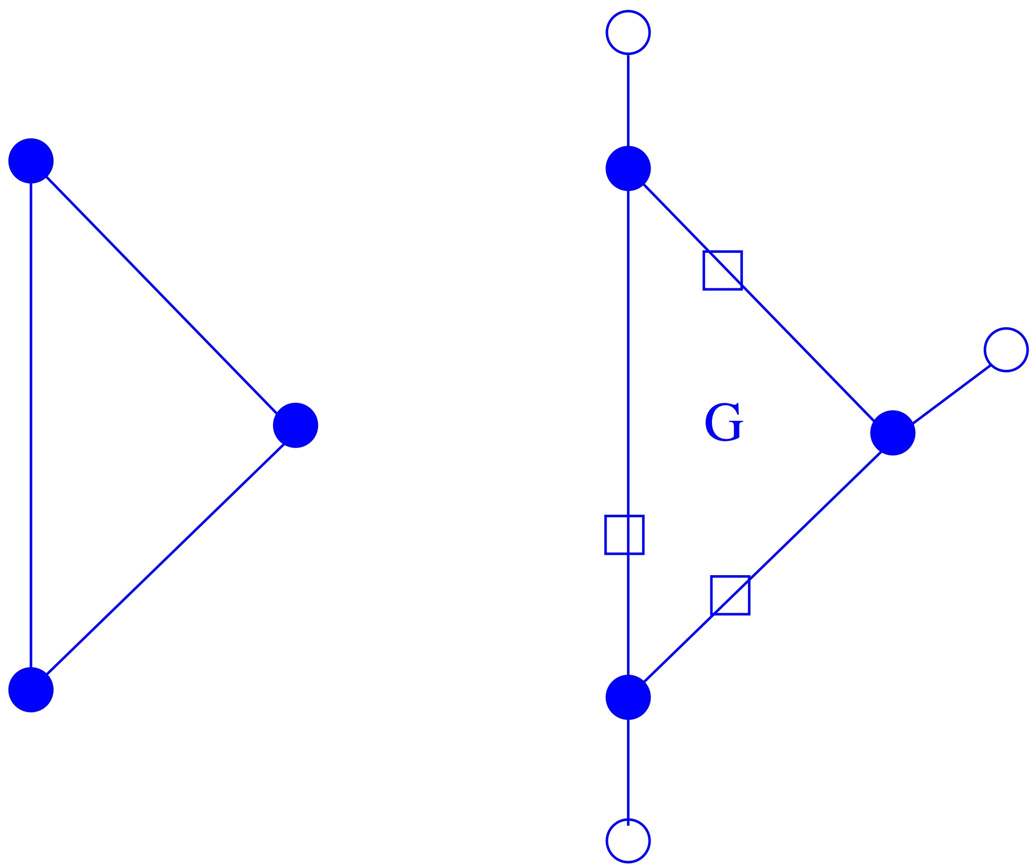
$$\rho_3(G)\gamma_{\text{pr}}(H) \leq \gamma_{\text{pr}}(G \Box H)\,.$$

Corollary: If G and H have no isolated vertices and $\gamma_{\text{pr}}(G) = 2\rho_3(G)$, then

$$\frac{1}{2} \cdot \gamma_{\text{pr}}(G)\gamma_{\text{pr}}(H) \leq \gamma_{\text{pr}}(G \Box H)\,,$$

and this bound is sharp.

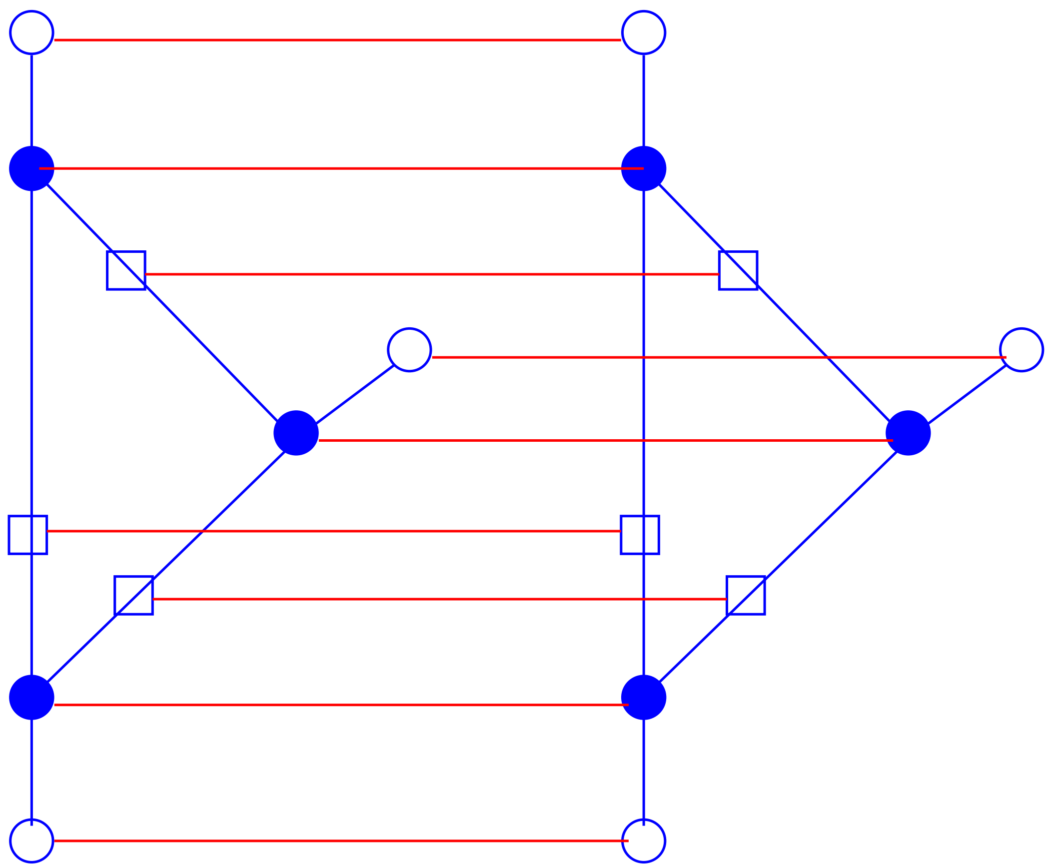
$$\frac{1}{2} \cdot \gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(K_2) \leq \gamma_{\mathrm{pr}}(G \Box K_2)$$



Subdivide each edge and add one leaf neighbor to each original vertex to get G.

$$\gamma_{\mathrm{pr}}(G) = 6 = 2 \cdot 3 = 2 \cdot \rho_3(G)$$

$\frac{1}{2} \cdot \gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(K_2) \leq \gamma_{\mathrm{pr}}(G \square K_2)$



Minimum paired–dominating set

$\frac{1}{2} \cdot \gamma_{\mathrm{pr}}(G)\gamma_{\mathrm{pr}}(K_2) = \frac{1}{2} \cdot 6 \cdot 2 = 6 = \gamma_{\mathrm{pr}}(G \square K_2)$

Summary
3-Packing & Paired-Domination on $\Box$

- $2\rho_3(G) \leq \gamma_{\mathrm{pr}}(G)$ for every graph G.

- $\rho_3(G)\gamma_{\mathrm{pr}}(H) \leq \gamma_{\mathrm{pr}}(G \Box H)$ for every H.

- $2\rho_3(T) = \gamma_{\mathrm{pr}}(T)$ for every tree T.

Open Problems and Questions

- Prove Vizing's Conjecture:
  $\gamma(G)\gamma(H) \leq \gamma(G \Box H)$.

- Improve the bound of Clark and Suen.
  That is, find a constant $k > \frac{1}{2}$ such that

$$\gamma(G \Box H) \geq k \cdot \gamma(G)\gamma(H)\,.$$

- Find the largest constant k such that for all G and H with no isolated vertices

$$k \cdot \gamma_t(G)\gamma_t(H) \leq \gamma_t(G \Box H).$$

$$\frac{1}{2}\gamma_t(G)\gamma_t(H) \leq \gamma_t(G \Box H)?$$

Open Problems and Questions

- Find other pairs of packing, ρ', and domination, γ', invariants together with a graph product, $\oplus$, such that
  - $\rho'(G) \leq \gamma'(G)$ for every graph G.
  - $\rho'(G)\gamma'(H) \leq \gamma'(G \oplus H)$ for every H.
  - $\rho'(T) = \gamma'(T)$ for every tree T.