On a multigraph complement and a generalization of Temperley's theorem to multigraphs

by

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Setting

$M = (V, E_M)$ – multigraph

$G = (V, E_G)$ – underlying graph

$E_M = \{ x^{(m_x)} \mid x \in E_G \}$

m_x = multiplicity of x in M

$\mathcal{T}_G$ = collection of spanning trees of G

$\mathcal{T}_M$ = collection of spanning trees of M

$$t(G) = |\mathcal{T}_G|, \qquad t(M) = |\mathcal{T}_M|$$

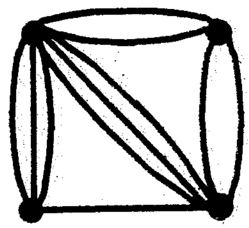
Observation

1. $\prod_{x \in \tau} (m_x) = \#$ copies of τ in M $(\tau \in \mathcal{T}_G)$

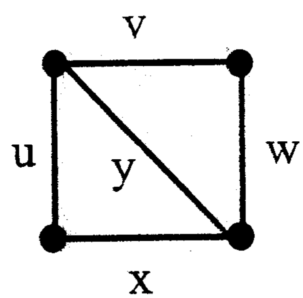
2. $t(M) = \sum_{\tau \in \mathcal{T}_G} \prod_{x \in \tau} (m_x)$

Example

M =



G =



$m_u = 3$

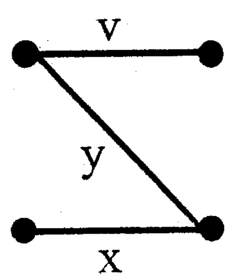
$m_v = 2$

$m_w = 2$

$m_x = 1$

$m_y = 3$

τ :

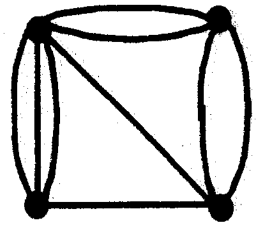
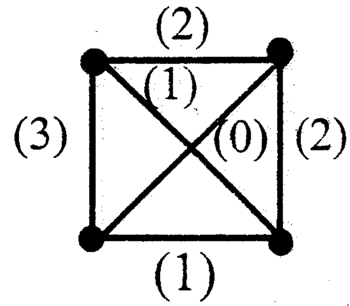


n_τ =(2) (3) (1) = 6 = # distinct sp. trees of M isomorphic to τ

Observation K_n = underlying graph of M

$$m_x \overset{\Delta}{=} 0 \text{ for each } x \in E(\overline{G})$$

Example

$M =$  $=$ 

Important subgraphs of K_n

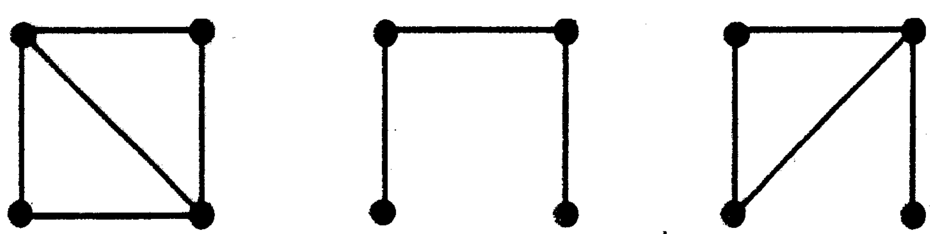
$$E(G) = \{x \in E(K_n) \mid m_x \geq 1\}$$

$$E(G') = \{x \in E(K_n) \mid m_x \geq 2\}$$

$$E(\tilde{G}) = \{x \in E(K_n) \mid m_x \neq 1\}$$

$$= E(G') \cup E(\overline{G})$$

Ex



G $\quad$ G' $\quad$ $\tilde{G}$

A "Temperley" Theorem

Given multigraph M, underlying graph G

and $\widetilde{G} = \{x \in K_n \mid m_x \neq 1\}$

Then

$$t(M) = \sum_{\substack{F \text{ sp forest} \\ \text{of } \widetilde{G}}} \left(\prod_{x \in F} (m_x - 1) \right) |A_F^C|$$

where $A_F^C = \{\tau \in \mathcal{T}_{K_n} \mid F \subseteq \tau\}$

<u>Lemma(Moon)</u> $|A_F^C| = n^{n-2-|F|} \upsilon(F)$

where $\upsilon(F)$ = product of orders of the comp. of F

Consequence

$$t(M) = \sum_{\substack{F \text{ sp forest} \\ \text{of } \widetilde{G}}} \left(\prod_{x \in F} (m_x - 1) \right) n^{n-2-|F|} \upsilon(F)$$

Special Case
(Temperley's Formula)

If M is a simple graph, i.e., M = G,

then $m_x \neq 1$ for $x \in E(K_n)$

iff

$x \in \overline{G}$

and in this case, $m_x = 0$. Thus

$$|\mathcal{T}_G| = \sum_{\substack{F \text{ sp forest} \\ \text{of } \overline{G}}} (-1)^{|F|} \left| A_F^C \right|$$

$$= n^{n-2} \sum_{\substack{F \text{ sp forest} \\ \text{of } \overline{G}}} \left(-\frac{1}{n} \right)^{|F|} \upsilon(F)$$

A Useful Construct
–A Multigraph Complement

Given:

Multigraph M=(V, E); underlying graph G

where $\mathbf{E} = \left\{ \mathbf{x}^{(\mathbf{m}_\mathbf{x})} \mid \mathbf{x} \in \mathbf{E}(\mathbf{K}_\mathbf{n}) \right\}$

and $\mathbf{m}_\mathbf{x} = \mathbf{0}$ **if** $\mathbf{x} \notin \mathbf{E}(\mathbf{G})$

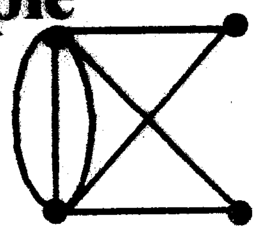
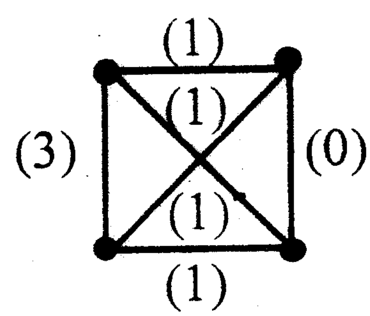
DEFINE

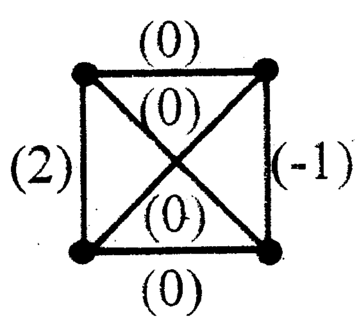
$\overline{\mathbf{M}} = (\mathbf{V}, \overline{\mathbf{E}})$

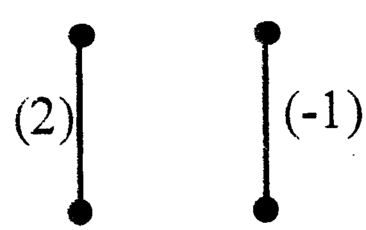
where $\overline{\mathbf{E}} = \left\{ \mathbf{x}^{(\mathbf{m}_\mathbf{x} - 1)} \mid \mathbf{x} \in \mathbf{E}(\mathbf{K}_\mathbf{n}) \right\}$

NOTATION $\overline{\mathbf{m}}_\mathbf{x} \stackrel{\Delta}{=} \mathbf{m}_\mathbf{x} - \mathbf{1}$

Example

M :  = 

$\overline{\mathbf{M}}$ = 

= 

$\tilde{\mathbf{G}}$ = **"underlying" graph of** $\overline{\mathbf{M}}$

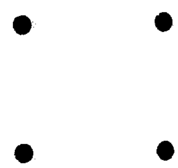
$t(M) =$

$$\sum_{\substack{F \text{ sp forest} \\ \text{of } \widetilde{G}}} \overline{m}_F \, n^{n-2-|F|} \upsilon(F)$$

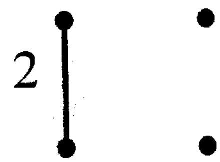
where $\overline{m}_F \stackrel{\Delta}{=} \prod_{x \in F} \overline{m}_x$, or 1 if $F = \Phi$

(Here F is just a sp forest $\widetilde{G}$ - the underlying graph of $\overline{M}$)

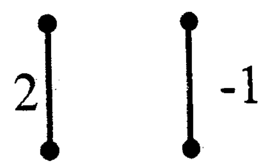
Example



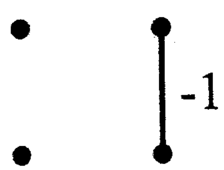
term (1)(4^2)



term (2)(8)



term (2)(-1)(4)



term (-1)(8)

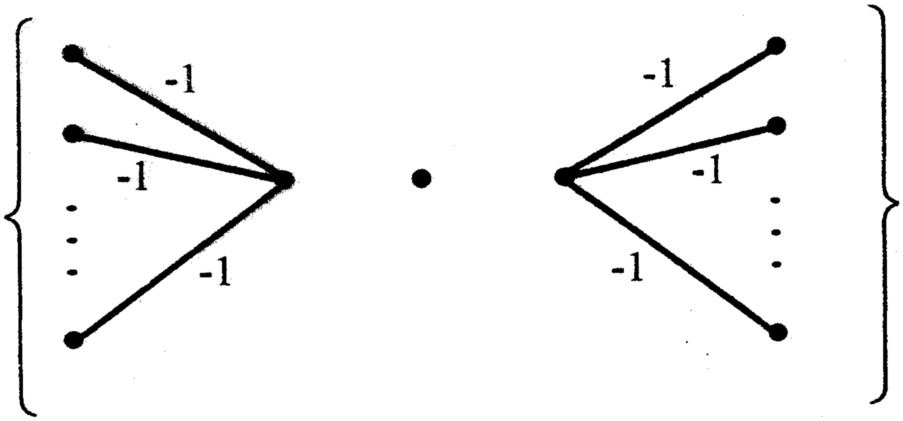
t(M) = 16

Applications

1. $\forall$ **odd** $n \geq 5$ $\exists$ **(simple) graph G and multigraph M having the same # of n nodes and e edges s.t.** $t(G) = t(M)$

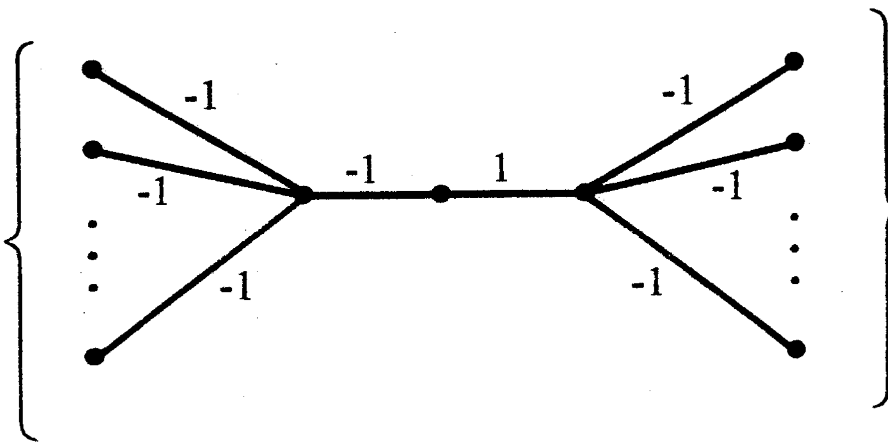
Proof : Let $n = 2\ell + 5$ and set

$\overline{G}$: ℓ



$\ell + 2 = k$

$\overline{M}$: ℓ



$\ell + 2 = k$

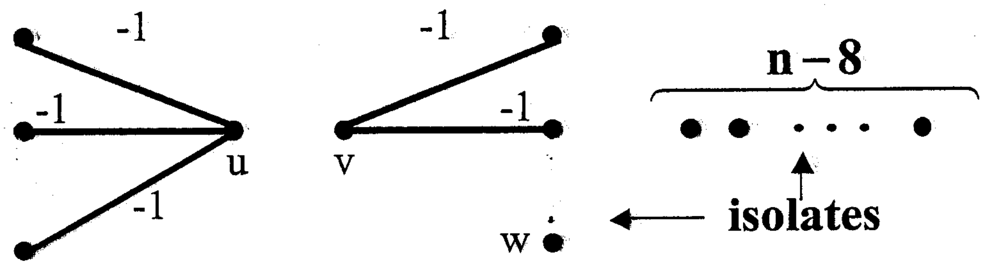
$$t(G) = t(M)$$

$$= n^{n-2}\left(1 - \frac{1}{n}\right)^{k+\ell}\left[\left(1 - \frac{k}{n-1}\right)\left(1 - \frac{\ell}{n-1}\right)\right]$$

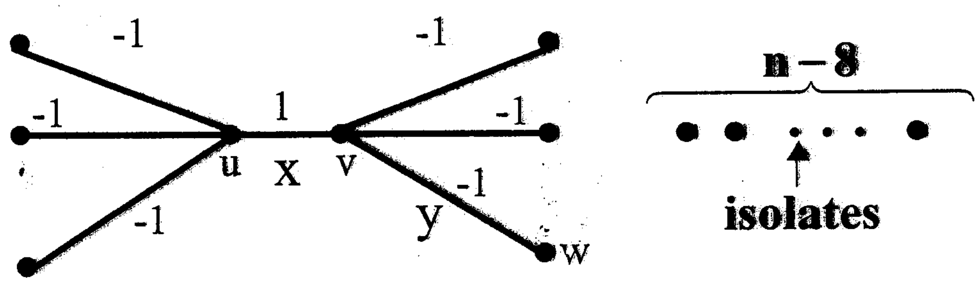
2. "Local" switches, i.e., removal of a multiedge and addition of a simple edge DO NOT work in GENERAL

Generic Example

$\overline{G}$:



$\overline{M}$



t(M) > t(G)

G obtained from M by moving one u – v edge to the vacant v – w position

Proof Idea–all terms in t(G) formula cancel with corresp. terms in t(M) expression

–remaining terms in t(M) expression

i.e., for $x \in F$, $y \notin F$; $x \notin F$, $y \in F$; $x \in F$, $y \in F$ sum to a positive #

Some other applications

- Some special "symmetric" switches work
  (i.e. increasing # sp trees)

- Tree formulas for special cases

- Count # r-cycles in a multigraph

- Formula for # of 1-factors in a multigraph

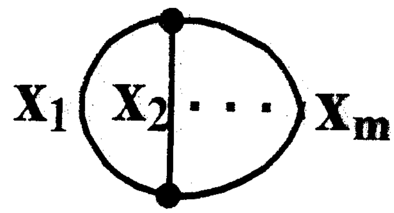
- Formula for # of Hamilton paths and
  Ham. Cycles in a multigraph

ETC

The transparencies to follow outline the proof of the main theorem in some detail

Stratifying M
(Labeling the edges of M)

$E_S \equiv \{ x_i \mid x \in E_G, 1 \leq i \leq m_x \}$



Note: x_i = x with label i

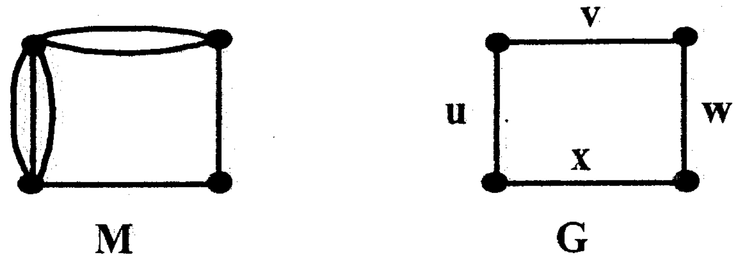
Observation

1. Given $\tau \in \mathcal{T}_G$, $\hat{\tau} = \{x_{i(x)} \mid x \in \tau, 1 \leq i(x) \leq m_x\} \in \mathcal{T}_M$ is a copy of τ (in M)

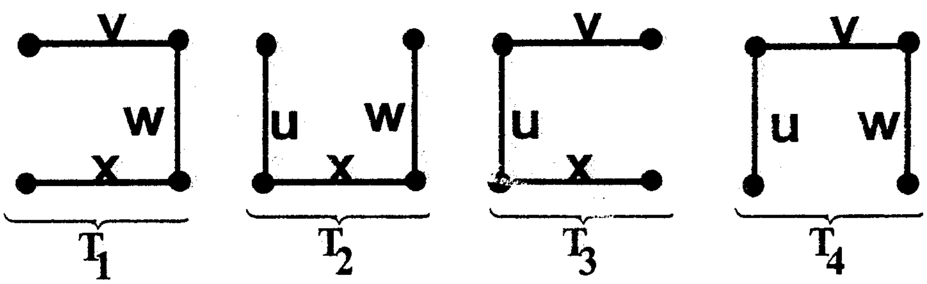
2. $\mathcal{T}_M$ = set of all $\hat{\tau}$'s =

$$\underbrace{\{\hat{\tau} \mid x \in \tau \Rightarrow i(x) = 1\}}_{"=" \ \mathcal{T}_G} \,\dot{\cup}\, \underbrace{\{\hat{\tau} \mid \exists\, x \in \tau \text{ s.t. } i(x) \geq 2\}}_{\mathcal{T}_M^2}$$

Example



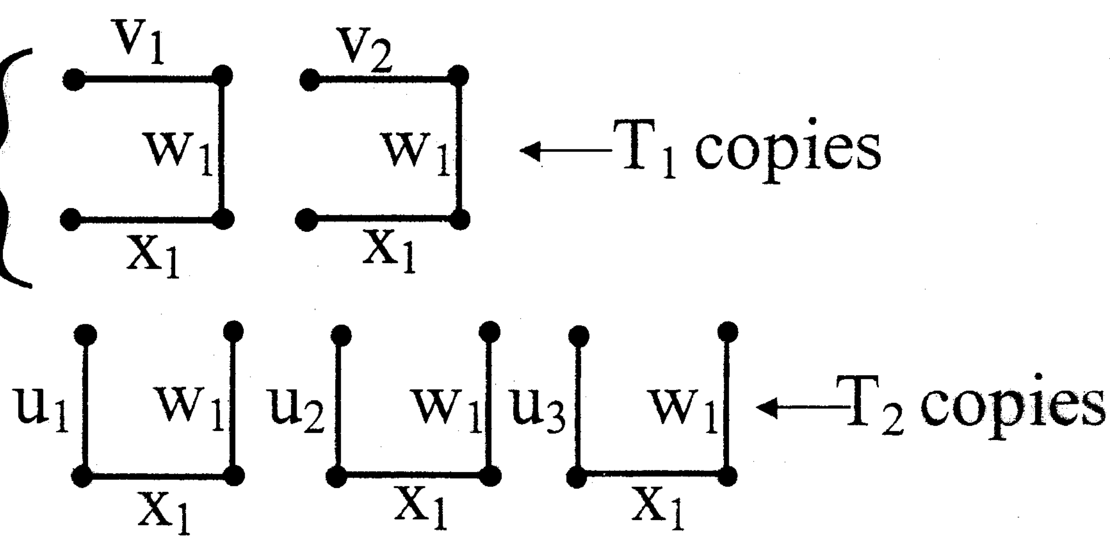
$\mathcal{T}_G = \{$



$\}$

$$E_S = \left\{ \underbrace{u_1,\ u_2,\ u_3}_{\text{copies of } u} \quad \underbrace{v_1,\ v_2}_{\text{copies of } v} \quad \underbrace{w_1}_{\text{copies of } w} \quad \underbrace{x_1}_{\text{copies of } x} \right\}$$

$\mathcal{T}_M = \{$



ETC

$\}$

$$\left|\mathcal{T}_M^2\right| = \left|\left\{\hat{\tau} \mid \exists\, x \in \tau \text{ s.t. } i(x) \geq 2\right\}\right|$$

<u>**IDEA**</u>: Partition $\mathcal{T}_M^2$ according to the set of edges having label ≥ 2

$$C_X \stackrel{\Delta}{=} \left\{\hat{\tau} \mid x \in \tau \text{ s.t. } i(x) \geq 2\right\}$$

all the labeled trees in which x appears with label ≥ 2

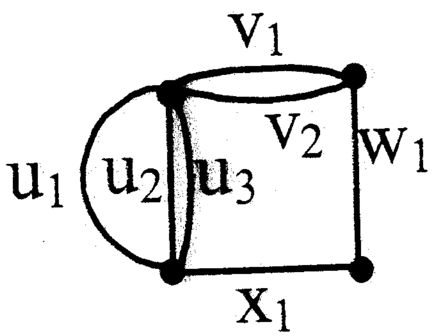
$$\sim C_X \stackrel{\Delta}{=} \left\{\hat{\tau} \mid x \in \tau \Rightarrow i(x) = 1\right\}$$

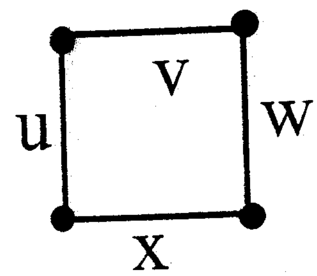
all the labeled trees in which

either : no copy of x exists,

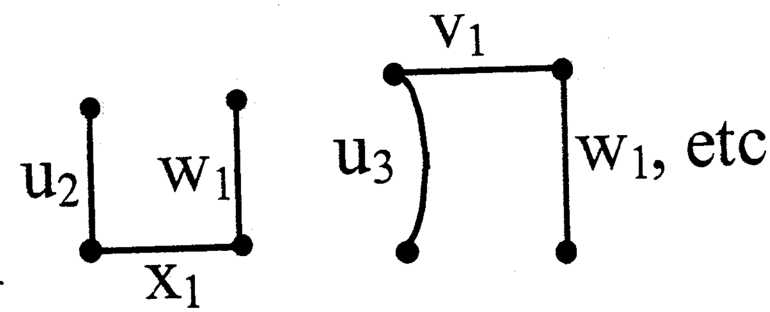
or $i(x) = 1$

Example





$C_u = \{\hat{\tau} \mid u_2 \text{ or } u_3 \in \hat{\tau}\} = \{$

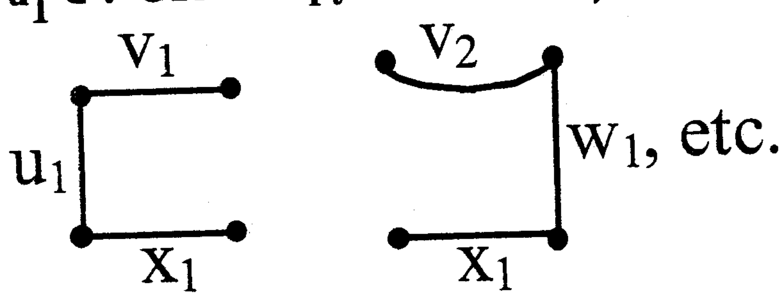


$\}$

$(|C_u| = 10)$

$\sim C_u = \{\hat{\tau} \mid u_1 \in \hat{\tau} \text{ OR no copy of u lies in } \hat{\tau}\}$

$= \{$



$\}$

$(|\sim C_u| = 7)$

$$I \subseteq E_G$$

$$C_I \stackrel{\Delta}{=} \left(\bigcap_{x \in I} C_x \right) \cap \left(\bigcap_{x \in E-I} \sim C_x \right)$$

Observations:

0. $\hat{\tau} \in C_I$ IFF

   – each $x \in I$ "appears" in $\hat{\tau}$ with label ≥ 2

   – each $x \in E_G - I$ either

   • No copy of x appears in $\hat{\tau}$

   OR • x appears in $\hat{\tau}$ with label $= 1$

1. $C_I \neq \Phi \Rightarrow I$ a sp. forest of G' where

   $E(G') = \{x \in E(G) \mid m_x \geq 2\}$

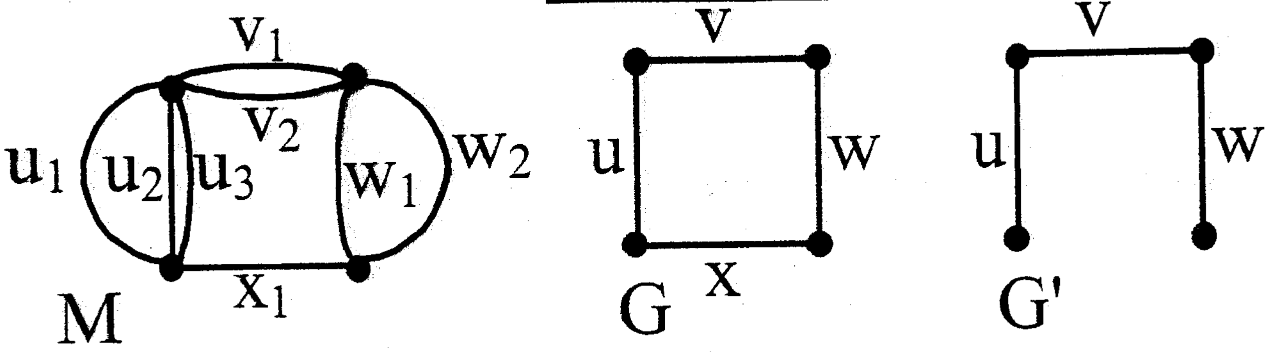
2. If $\tau \in \mathcal{T}_G$, then τ contributes to C_I IFF $I \subseteq \tau$

3. If $I \subseteq \tau$, then τ contributes

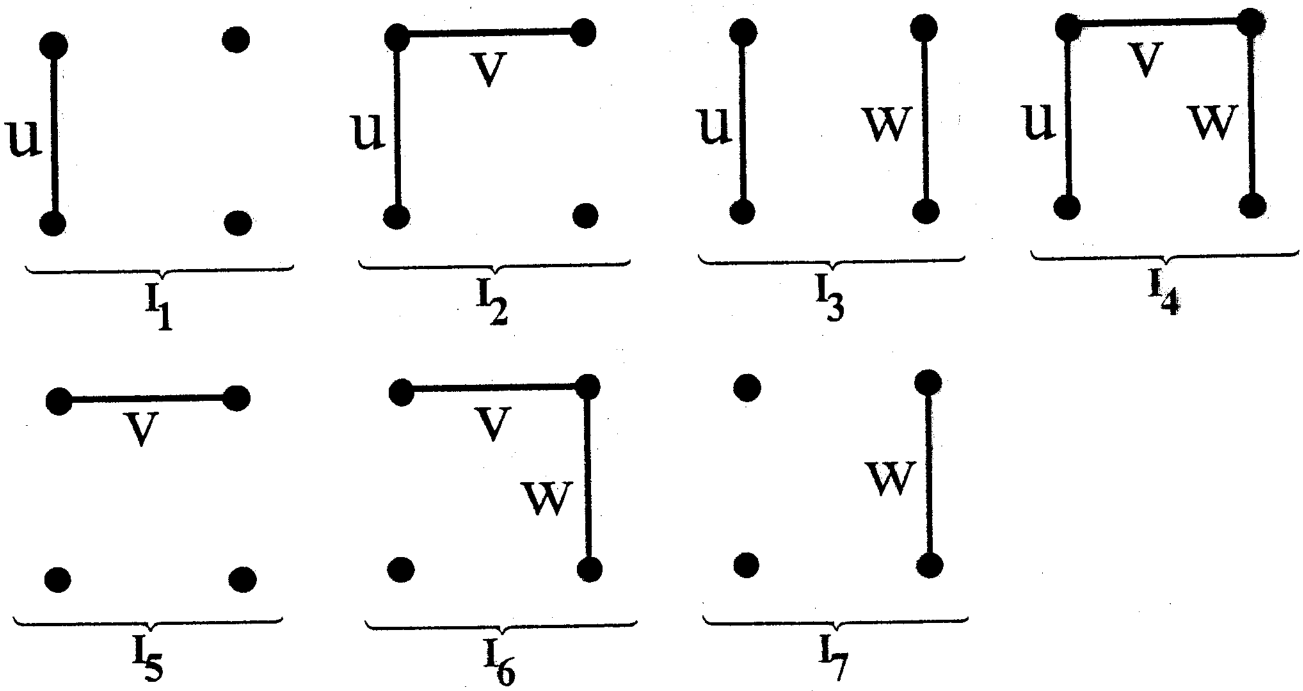
   $\prod_{x \in I} (m_x - 1)$ $\hat{\tau}$'s to C_I

Conclusion $|C_I| = \left(\prod_{x \in I} (m_x - 1) \right) \underbrace{\{\tau \in \mathcal{T}_G \mid \tau \supseteq I\}}_{A_I}$

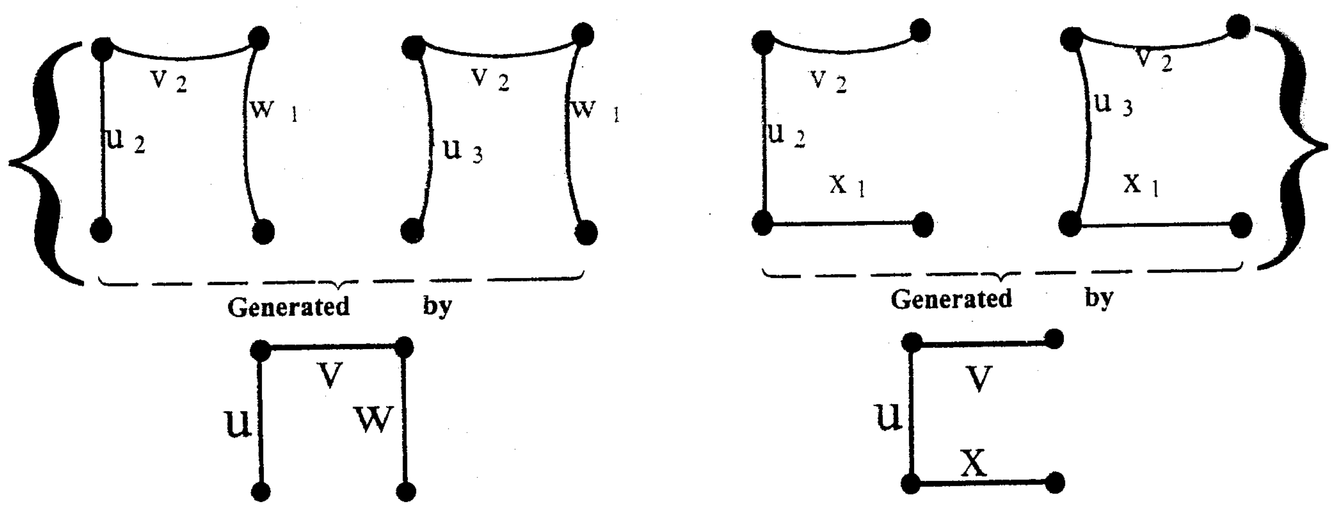
Example



NON EMPTY I's



$C_{\{u,v\}}=$



THEOREM $\left|\mathcal{T}_M^2\right| = \sum_{\Phi \neq I \text{ sp frst of } G'} \prod_{x \in I} (m_x - 1) \left|A_I\right|$

PROOF $\mathcal{T}_M^2 = \dot{\bigcup}_{\Phi \neq I \text{ sp frst of } G'} C_I$

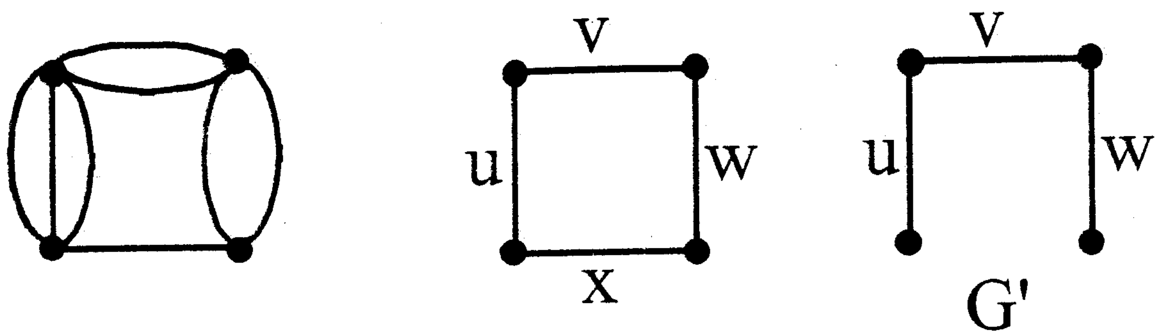
CONVENTIONS: A_Φ "=" $\mathcal{T}_G$

$$\prod_{x \in \Phi} (m_x - 1) \stackrel{\Delta}{=} 1$$

COROLLARY $t(M) = \sum_{I \text{ sp frst of } G'} \prod_{x \in I} (m_x - 1) \left|A_I\right|$

PROOF $\mathcal{T}_M$ "=" $\mathcal{T} \bigcup \mathcal{T}_M^2$

Example



- $|A_{\{u\}}| = |A_{\{v\}}| = |A_{\{w\}}| = 3$

•• $|A_{\{u,v\}}| = |A_{\{u,w\}}| = |A_{\{v,w\}}| = 2$

••• $|A_{\{u,v,w\}}| = 1$

$t(M) = 4 +$

• $2(3) + 1(3) + 1(3) +$

•• $[(2)(1)](2) + [(2)(1)](2) + [(1)(1)](2) +$

••• $[(2)(1)(1)]$

$= \underline{28}$

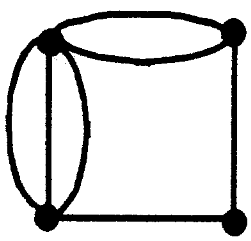
In Preparation for a "Temperley" Theorem

Given: multigraph M, underlying graph G
Define: multigraph M^c with underlying graph K_n as follows –

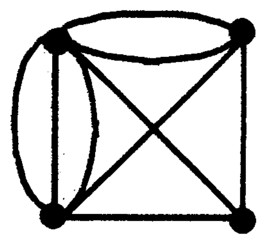
$$x \in E(G), \quad m_x^c = m_x \ (\text{in } M)$$

$$x \in E(\overline{G}), \quad m_x^c = 1$$

Example



M



M^c

Observation :

$$t(M^c) = \sum_{F \text{ sp frst of } G'} \prod_{x \in F} (m_x - 1) \left| A_F^c \right|$$

where $A_F^c = \{\tau \mid \tau \in \mathcal{T}_{K_n}, \ F \subseteq \tau\}$

Remark:

$t(M^c)$ counts sp trees that are not in $\mathcal{T}_M$; i.e. those that include at least one x, where $x \in E(\overline{G})$

Counting the "bad" trees

$$B \overset{\Delta}{=} \left\{ \hat{\tau} \in \mathcal{T}_{M^c} \,\middle|\, \tau \cap E(\overline{G}) \neq \Phi \right\}$$

For F a sp forest of G'

$$B_F = \left\{ \hat{\tau} \in \mathcal{T}_{M^c} \,\middle|\, \hat{\tau} \in C_F, \tau \cap E(\overline{G}) \neq \Phi \right\}$$

Then

$$B = \dot{\bigcup_{F}} B_F$$

with $E(\overline{G}) = \left\{ b_1, b_2, \cdots, b_{e(\overline{G})} \right\}$

$$B_F = \bigcup_{j=1}^{e(\overline{G})} \left\{ \hat{\tau} \in \mathcal{T}_{M^c} \,\middle|\, \hat{\tau} \in C_F, \ b_k \in \tau \right\}$$

Inclusion/ Exclusion$\Rightarrow$

$$|B| = - \sum_{\substack{F \text{ sp forest} \\ \text{of } G'}} \ \sum_{\substack{\Phi \neq K \text{ sp forest} \\ \text{of } \overline{G}}} \ \prod_{x \in F \cup K} (m_x - 1) \left| A^c_{F \cup K} \right|$$

Conclusion

$$t(M) = t(M^{c}) - |B|$$

$$= \sum_{\substack{F \text{ sp forest} \\ \text{of } \tilde{G}}} \prod_{x \in F} (m_x - 1) \left| A_F^{c} \right|$$