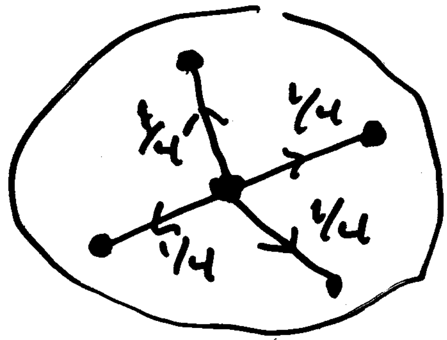
Richard Duke

JOINT WORK WITH ERIC BUSSIAN

RANDOM WALKS ON GRAPHS



DISCRETE-TIME MARKOV CHAINS

MOON ('73) - WALKS ON TREES

DOYLE, SNELL ('79, '84)
CORRESPONDENCE WITH RESISTANCES IN ELECTRICAL CIRCUITS

APPLICATIONS TO ALGORITHMS AND COMPLEXITY QUESTIONS

COVERING TIME

$C(G) =$ MAX OVER STARTING POINTS OF EXPECTED TIME TO VISIT EVERY VERTEX

ALEUNIAS ET AL. ('79)

$$C(G) = O(|V||E|) = O(n^3) \quad (|V| = n)$$

E.g. $C(K_n)$ — "COUPON COLLECTOR PROBLEM"

AFTER VISITING j VERTICES, THE PROBABILITY OF REACHING A NEW VERTEX IS $\frac{n-j}{n-1}$.

EXPECTED TIME TO REACH A NEW VERTEX IS $\frac{n-1}{n-j}$.

$$C(K_n) = (n-1)\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right)$$
$$\sim (n-1) \ln n$$

FEIGE ('93)

$$C(G) \geq (1+o(1))\, n \ln n$$

FOR EVERY GRAPH G

HITTING TIME

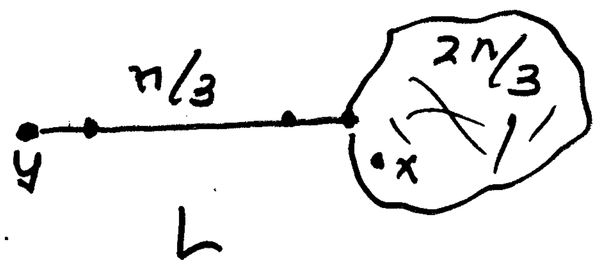
$H(x,y)$ – EXPECTED TIME TO REACH y FROM x

$$H(G) - \max_{(x,y)} H(x,y)$$

$$H(G) \leq C(G) = O(n^3).$$

BRIGHTWELL, WINKLER ('90)

$$H(L) = \frac{4}{27}n^3 + o(n^3)$$



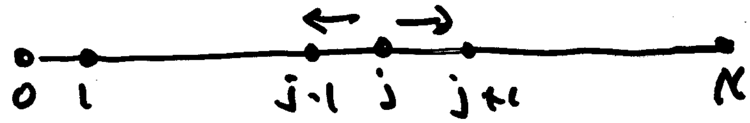
$$\text{SO } C(L) \geq \frac{4}{27}n^3 + o(n^3)$$

$$\text{FEIGE ('95)} \quad \max C(G) = \frac{4}{27}n^3 + o(n^3)$$

(EVEN FOR "CYCLIC COVER TIME")

Walks on Trees

"Gambler's Ruin"

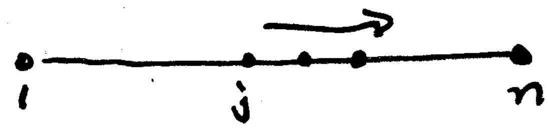


a_j = Prob. of "ruin" – reaching 0 before N, starting from j.

$$a_0 = 1, \quad a_N = 0 \qquad a_j = \frac{1}{2} a_{j-1} + \frac{1}{2} a_{j+1}$$

$$\underline{a_j = 1 - j/N}$$

Hitting time



$a_j = H(j,n)$ (1 no longer "absorbing")

$$a_1 = a_2 + 1 \qquad a_n = 0$$

$$a_j = \frac{1}{2}(a_{j-1}+1) + \frac{1}{2}(a_{j+1}+1)$$

$$a_j = (n-1)^2 - (j-1)^2$$

So $\underline{a_1 = (n-1)^2}$

$H(1, n) = (n-1)^2 \Rightarrow C_1(P) = (n-1)^2$

FROM 1

SIMILARLY, COVERING FROM j

$$C_j(P) = \frac{5}{4}(n-1)^2 - \left(j - \frac{n+1}{2}\right)^2$$

MAX $= \frac{5}{4}(n-1)^2$ FROM THE CENTER(S)

BRIGHTWELL, WINKLER ('90)

<u>CONJECTURE</u>: MAX $C(T)$ FOR ALL TREES $= \frac{5}{4}(n-1)^2$

PROVED BY FEIGE ('97)

COVERING EDGES

FOR A PATH, FROM 1, $H(1,n) = (n-1)^2$
IMPLIES $(n-1)^2$ TO TRAVERSE EACH EDGE IN AT LEAST ONE DIRECTION

$CE(G)$ – EXPECTED TIME TO TRAVERSE EACH EDGE IN BOTH DIRECTIONS.

E.g. $CE(K_n) = \Theta(n^2 \ln n)$

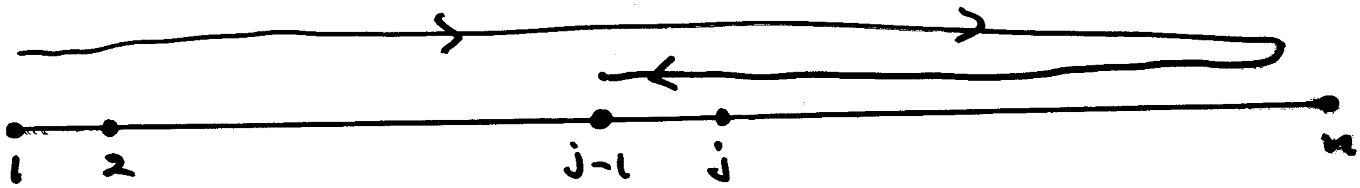
ZUCKERMAN ('91)

$$CE(G) = O(|V||E|) = O(n^3)$$

PATH?

$$CE(P) \leq H(1,n) + H(n,1) = 2(n-1)^2$$

IS IT THIS LARGE?



HOW FAR BACK IS THE 1ST EDGE $(j-1, j)$ NOT TRAVERSED $j \to j-1$?

$P(n,j)$ — PROB. THAT j IS THE LEAST SUCH VALUE.

$$CE_1(P) = H(1,n) + \sum_{2 \le j \le n} P(n,j) H(n,j)$$

$$= (n-1)^2 + \sum_{2 \le j \le n} (n-j+1)^2 P(n,j)$$

$$P(n,j) = ?$$

$$\sum_{2 \le j \le n} (n-j+1)^2 P(n,j) = ?$$

cn^2?

(1) FOR $2 \leq j \leq n-1$ $\quad P(n,j) = \frac{n-j}{n-j+1} P(n-1, j)$

(2) FOR $j = n$ $\quad P(n,n) = \sum_{2 \leq j \leq n-1} \frac{1}{n-j+1} P(n-1, j)$

ITERATING (1) FROM $h = j+1$ TO n

YIELDS

(3) $$P(n,j) = \frac{1}{n-j+1} P(j,j)$$

ONLY NEED THE $P(j,j)$ – THE LAST EDGE!

LET $f_{j-1} = P(j,j)$

$\sum_{2 \leq j \leq n} P(n,j) = 1$ AND (3) YIELD

$$\underline{\sum_{1 \leq j \leq n-1} \frac{1}{n-j} f_j = 1}$$

$f_1 = 1$, $f_2 = 1/2$, $f_3 = 5/12$, $f_4 = 3/8$

$$f_9 = \frac{1,070,017}{3,628,800}$$

$$\left(\sum_{1 \leq j \leq n} f_j h_{n-j} = n \right)$$

$$CE_1(P) = (n-1)^2 + \sum_{1 \le j \le n-1} (n-j) f_j$$

LET $g_{n-1} = \sum_{1 \le j \le n-1} (n-j) f_j \quad (g_0 \equiv 0)$

ORDINARY GENERATING FUNCTIONS

$$F(x) = \sum_{j \geq 0} f_j x^j$$

$$G(x) = \sum_{j \geq 0} g_j x^j$$

$\sum_{1 \le j \le n-1} \frac{1}{n-j} f_j = 1$ IMPLIES

$$F(x) \cdot (-\ln(1-x)) = (x^2 + x^3 + \cdots) = \frac{x^2}{1-x}$$

$$F(x) = \frac{-x^2}{(1-x)\ln(1-x)}$$

$$g_{n-1} = \sum_{1 \le j \le n-1} (n-j) f_j$$

$$G(x) = F(x) \cdot \frac{1}{(1-x)^2} = \frac{-x^3}{(1-x)^3 \ln(1-x)}$$

$$\frac{1}{-\ln(1-x)} = ?$$

$$-\ln(1-x) = x + x^2/2 + x^3/3 + \cdots$$

HAS NO RECIPROCAL - SINCE CONSTANT TERM $= 0$.

$$\frac{1}{\frac{-\ln(1-x)}{x}} = \frac{-x}{\ln(1-x)} \quad \text{DOES EXIST!}$$

LET $Q(x) = \sum_{j \geq 0} q_j x^j = \dfrac{x}{\ln(1+x)}$

THEN
$$F(x) = \frac{x}{1-x} Q(-x)$$
$$G(x) = \frac{x^2}{(1-x)^3} Q(-x)$$

$$\boxed{Q(x) = \frac{x}{\ln(1+x)} = ?}$$

HIGHER ORDER BERNOULLI NUMBERS

(GREGORY, 1670), (L. CARLITZ, '52-'67) (KNUTH '62)

NEWTON'S INTERPOLATION FORMULA

$$P(x) = P(0)\binom{x}{0} + \Delta P(0)\binom{x}{1} + \cdots + \Delta^n P(0)\binom{x}{n},$$

WHERE $\binom{x}{n} = \frac{1}{k!}(x)(x-1)\cdots(x-k+1)$

CONSIDER POLYNOMIALS $P_n(x)$
SUCH THAT

$$D(P_n(x)) = \binom{x}{n-1} \qquad (1)$$

SO,
$$\Delta(P_n(x)) = P_{n-1}(x) \qquad (2)$$

$$P_n(x) = b_0\binom{x}{n} + b_1\binom{x}{n-1} + \cdots + b_n\binom{x}{0}$$

(2) IMPLIES THE b_j ARE INDEPENDENT OF n.

$$\int_0^1 \binom{x}{n} dx = P_{n+1}(1) - P_{n+1}(0) = b_n$$

"BERNOULLI NUMBERS OF THE 2nd KIND"

Bernoulli Numbers

$$B_0 = 1, \quad \sum_{0 \le k \le n-1} B_k = 0, \quad \text{for } n > 1$$

$$B_0 = 1, \; B_1 = -1/2, \; B_2 = 1/6, \; B_3 = 0$$

$$B_{2k+1} = 0, \; k \geq 1 \quad ; \quad B_{2k} = ?$$

Bernoulli Polynomials

$$\mathcal{B}_n(x) = \sum_{k \geq 0} \binom{n}{k} B_k \, x^{n-k}$$

$$\int_0^{m+1} \mathcal{B}_n(x)\,dx = \sum_{0 \le k \le m} k^n$$

Euler's Summation Formula

$$\varphi_n(x) = \frac{1}{n!} \mathcal{B}_n(x)$$

$$D\,\varphi_n(x) = \varphi_{n-1}(x)$$

$$\Delta\,\varphi_n(x) = \frac{x^{n-1}}{(n-1)!}$$

$$\sum_{0 \le k \le n-1} \frac{(-1)^k}{n-k} b_k = 0 \qquad (b_0 = 1)$$

$$b_0 = 1, \quad b_1 = 1/2, \quad b_2 = -1/12, \quad b_3 = 1/24$$

$$b_9 = \frac{57{,}281}{7{,}257{,}600}$$

$$\sum_{n \ge 0} b_n x^n = \sum_{n \ge 0} \left(\int_0^1 \binom{t}{n} dt \right) x^n$$

$$= \int_0^1 (1+x)^t dt = \left[\frac{1}{\ln(1+x)} (1+x)^t \right]_{t=0}^{t=1}$$

$$= \frac{x}{\ln(1+x)}$$

$$Q(x) = \sum_{n \ge 0} b_n x^n = \frac{x}{\ln(1+x)}$$

$\frac{1-x}{x} F(x) = Q(-x)$ YIELDS

$$f_{n+1} - f_n = (-1)^n b_n = (-1)^n \int_0^1 \binom{x}{n} dx$$

$\binom{x}{n} = \binom{x-1}{n} + \binom{x-1}{n-1}$ LEADS TO

$$f_n = (-1)^{n-1} \int_0^1 \binom{x-1}{n-1} dx$$

So $\quad f_n < \left(1 - \frac{1}{2n}\right) \frac{1}{h_n}$

$$f_n \sim (1 + o(1)) \frac{1}{h_n}$$

$$G(x) = \frac{x}{(1-x)^3} Q(-x) = \frac{x}{(1-x)^3} \left(\sum_{k \geq 0} (-1)^k b_k x^k \right)$$

$$g_n = \int_0^1 \left(\sum_{0 \leq k \leq n-1} (-1)^{n-k+1} \binom{3+k-1}{k} \binom{t}{n-k-1} \right) dt$$

$$= \int_0^1 (-1)^{n-1} \binom{t-3}{n-1} dt$$

$$g_{n-1} = (-1)^{n-2} \int_0^1 \left(\frac{1}{2}\right)\left(\frac{t-3}{3}\right)\left(\frac{t-4}{4}\right) \cdots \left(\frac{t-(n-2)}{n-2}\right) \cdot$$
$$(t-(n-1)(n-2))\, dt$$

USING $(1-x) < e^{-x}$

$$g_{n-1} < \left(\frac{1}{2}\right) \int_0^1 e^{-t\left(\frac{1}{3}+\frac{1}{4}+\cdots\frac{1}{n-2}\right)} \left(t^2-(2n-1)t+n(n-1)\right) dt$$

INTEGRATION BY PARTS

$$g_{n-1} < \frac{n^2}{2(h_{n-2}-3/2)} \left(1 - \frac{1}{e^{h_{n-2}-3/2}}\right) + o(1)$$

$$g_{n-1} = O(n^2/\ln n)$$

$$g_{n-1} = \Theta(n^2/\ln n)$$

FROM OTHER STARTING POINTS

MAX FROM CENTER(S) FOR $n \geq 8$

(STILL $5/4(n-1)^2 + O(n^2/\ln n)$)

QUICKER FROM 2 AND $n-1$ THAN

FROM 1 OR n.

MINIMUM FOR TREES — STAR

CYCLE: $\binom{n}{2} + \Theta(n^2/\ln n)$