Finding Long Cycles in 3-connected Graphs

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Definition 1. *The circumference* $c(G)$ *of a graph G is the length of a longest cycle of G.*

The problem of approximating the circumferences of graphs is NP-hard .

Fact 2. *For most canonical NP-hard problem:*
Either dramatically improved approximation algorithms have been devised or strong negative results have been established.

Fact 3. *For finding long cycles, not much is known, positive or negative.*

- There is no known algorithm which guarantees an approximation ratio better than $n/\text{polylog}(n)$, and
- There is no hardness of approximation results that explains this situation

Special Classes of Graphs

1. Graphs with bounded degrees
2. Planar graphs
3. Graphs with forbidden minors
4. Graphs with large degrees
5. Graphs with forbidden subgraphs

Conjecture 4 (Bondy and Simonovits - 80). *There exists a constant* $0 < \beta < 1$ *such that* $c(G) \geq n^{\beta}$ *for any 3-connected cubic graph* G.

Jackson(-86) established the conjecture.

Theorem 5 (Feder, Motwani, and Subi -02). *There is a polynomial time algorithm for finding a cycle of length at least* $n^{\log_3 2}$ *in a 3-connected cubic graph.*

Theorem 6 (Jackson and Wormald-93). *If G is a 3-connected graph with maximum degree at most d, then $c(G) \geq \frac{1}{2}n^{\log_b 2} + 1$, where $b = 6d^2$.*

Their argument is technical and they did not address the algorithmic issue. In fact, a straightforward implementation of their argument is exponential.

Theorem **7** (C, Xu, and Yu- 03). *There is a cubic polynomial time algorithm for finding a cycle of length at least* $n^{\log_b 2}$ *in a 3-connected graph with maximum degree at most* d, *where* $b = 2(d-1)^2 + 1$.

Theorem **8** (C, Yu, and Zang-??). *There is a quadratic polynomial time algorithm for finding a cycle of length at least* $n^{\log_b 2}$ *in a 3-connected graph with maximum degree at most* d, *where* $b = 8(d-1) + 1$.

Conjecture 9 (Jackson and Wormald-93). *There is a function* $\alpha(d) > 0$ *such that* $c(G) \geq \alpha(d)n^{\log_{d-1} 2}$ *for any* 3*-connected graph with maximum degree at most* d, *where* $d \geq 4$.

Conjecture 10 (Jackson and Wormald-93). *There is a function* $\beta(d) > 0$ *such that* $c(G) \geq \beta(d)n$ *for any* 4*-connected graph with maximum degree at most* d, *where* $d \geq 4$.

Conjecture 11. *There is a function $\alpha(k) > 0$ such that $c(G) \geq \alpha(k)n$ for every k-connected k-regular graph G.*

Theorem 12 (Dirac)**.** *If G is a 2-connected graph with $\delta(G) \geq d$, then $c(G) \geq \min\{2d, n\}$.*

Theorem 13 (Fan - 86?)**.** *If G is a 3-connected k-regular graph, then $c(G) \geq \min\{3k, n\}$.*

Fact 14. *Every hamiltonian plane graph is 4-face colorable.*

Conjecture 15 (Tait -1886)**.** *Every 3-connected cubic planar graph is hamiltonian.*

Theorem 16 (Tutte - 56)**.** *There exists 3-connected cubic planar graph which is not hamiltonian.*

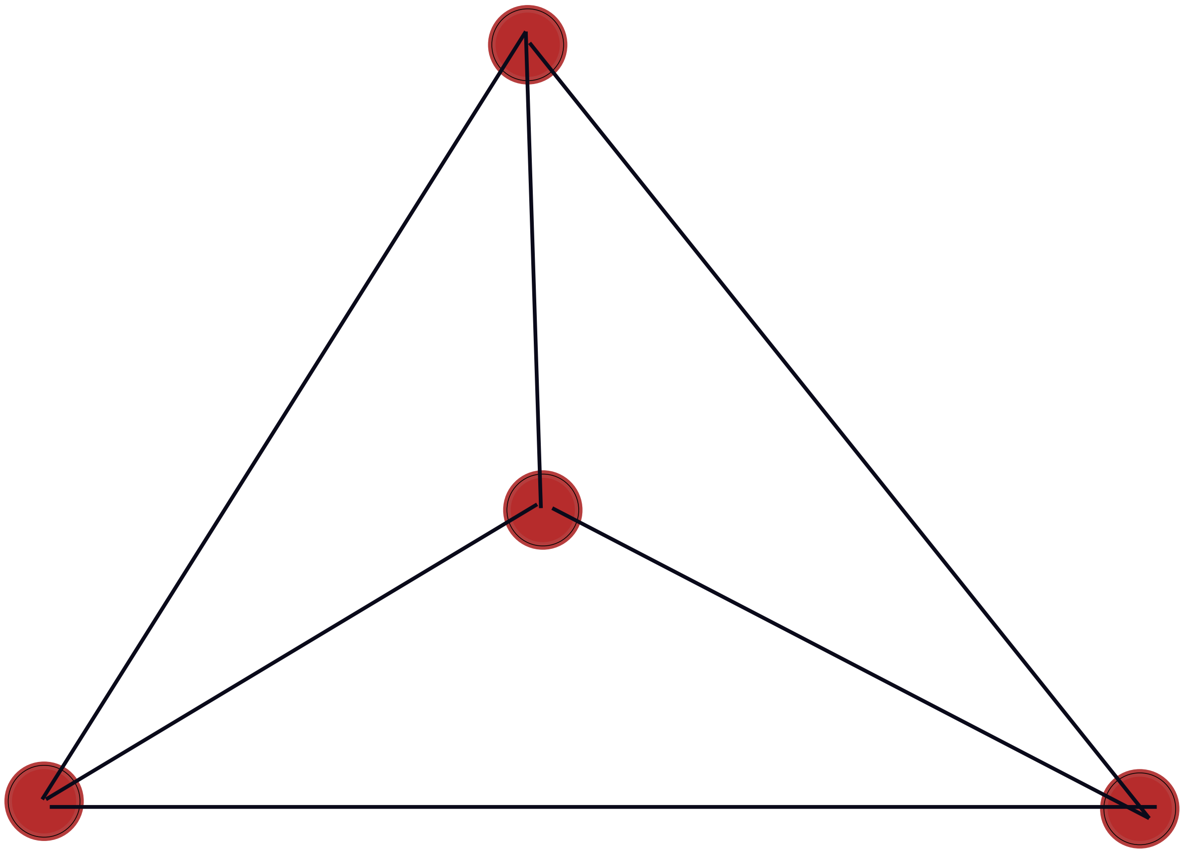
Theorem 17 (Whitney-32). *Every 4-connected plane triangulation contains a Hamiltonian cycle.*

Theorem 18 (Tutte - 56). *All 4-connected planar graphs have a hamiltonian cycle.*

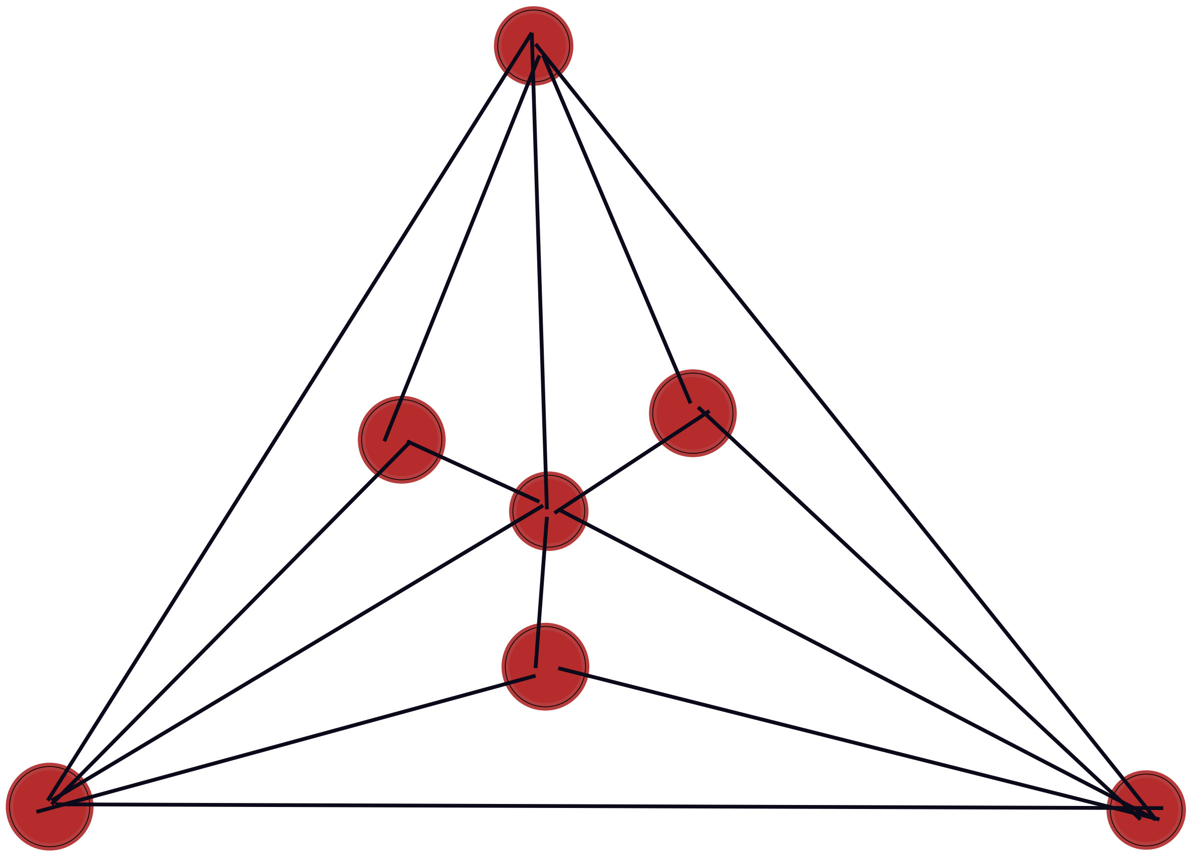
Conjecture 19 (Moon and Moser -63). *There is a universal constant $\alpha > 0$ such that $c(G) \geq \alpha n^{\log_3 2}$ for every 3-connected planar graph G*

Conjecture 20 (Grünbaum and Walther -73). *The same is true for 3-connected* **cubic** *planar graphs.*

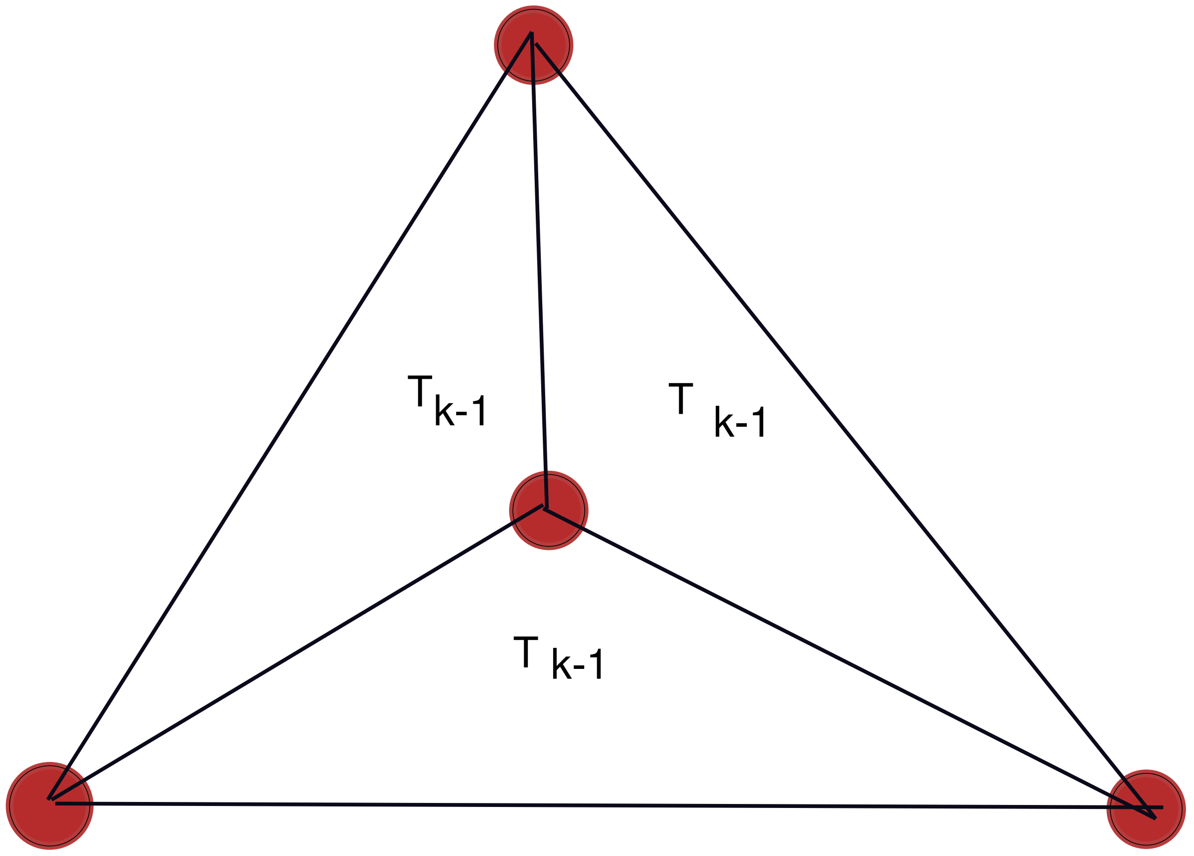
The exponent $\log_3 2$ is the best possible.



T_1



T_2



T_k

Fact 21. $c(T_k) < \frac{7}{2}n^{\log_3 2}$, *where* $n = |T_k|$.

Let G be a 3-connected planar graph of order n.

- (Barnette -66) $c(G) \geq \sqrt{\lg n}$.
- (Clark -85) $c(G) \geq e^{\sqrt{\lg n}}$.
- (Jackson and Wormald -92) $c(G) \geq \beta n^{\alpha}$, where β is some constant and $\alpha \approx 0.207$.
- (Gao and Yu -97) $c(G) \geq \beta n^{\alpha}$, where β is some constant and $\alpha \approx 0.4$.
- (Chung -??) improved α to 0.5.

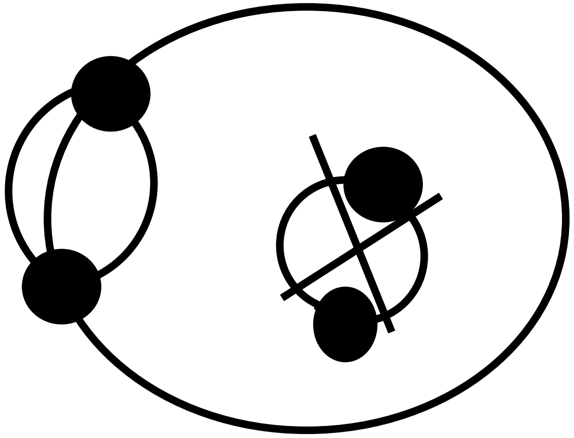
Theorem 22 (C, Yu-02). *Let G be a 3-connected graph with n vertices, and suppose that G is embeddable in the sphere, or the projective plane, or the torus, or the Klein bottle. Then $circ(G) \geq \Omega(n^{\log_3 2})$.*

Lemma 23. *Let m, n, k be non-negative real numbers. Then*

(1) $m^r + n^r \geq (m+n)^r$ *for* $0 < r < 1$.

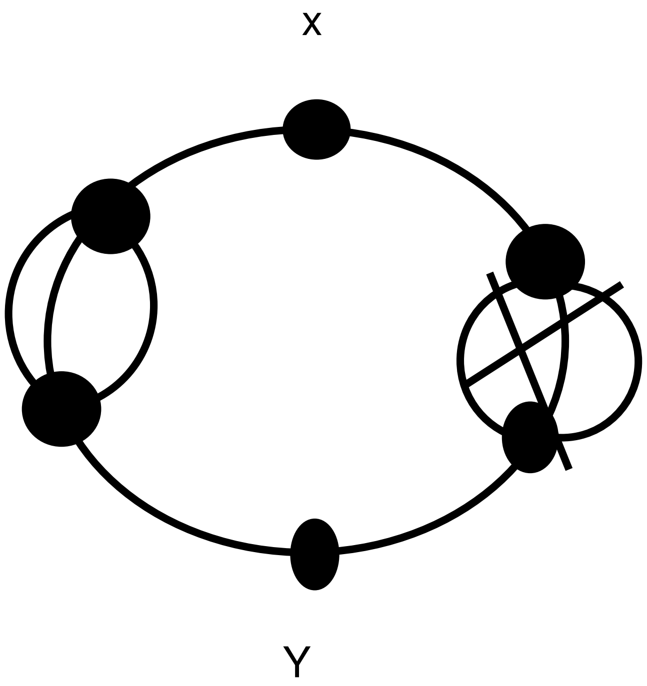
(2) $m^r + n^r \geq (m+n+k)^r$ *if* $0 \leq r \leq \log_3 2$ *and* $k = \min\{m, n, k\}$.

Definition 24. *A circuit graph is a pair (G, C), where G is a 2-connected plane graph and C is a facial cycle of G, such that, for any 2-cut S of G, every component of $G - S$ contains a vertex of C.*



Circuit Graph

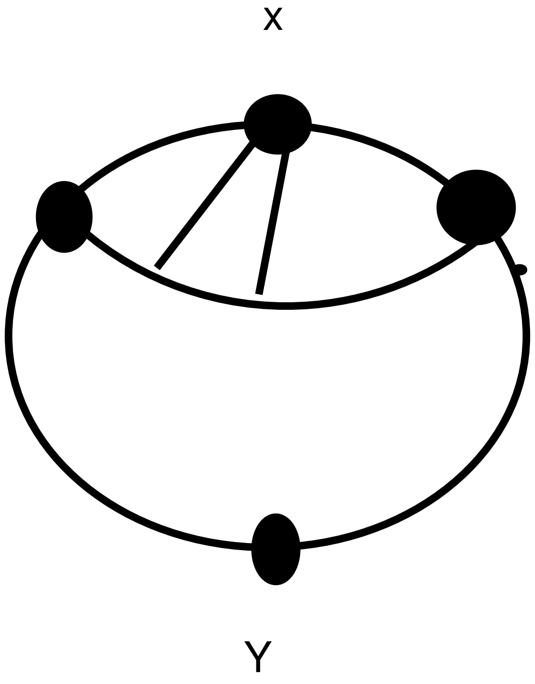
Definition 25. *Let (G, C) be a circuit graph, and let $x, y \in V(C)$. We say that (G, xCy) is a strong circuit graph if, for any 2-cut S of G, $S \cap V(yCx - \{x, y\}) \neq \emptyset$.*



Strong Circuit Graph

Theorem 26. *Let* (G, xCy) *be a strong circuit graph, and let* $w : V(G) \rightarrow \mathbb{R}^+$. *Then* G *contains an* $x - y$ *path* P *such that*

$$\Sigma_{v \in V(P-y)} [w(v)]^{\log_3 2} \geq [w(G-y)]^{\log_3 2}.$$



Highlight the proof

Theorem 27 (Böhme, Mohar, and C. Thomassen -02). *If G is a 3-connected graph on n vertices and of orientable genus g, then $c(G) \geq \epsilon(g)n^{\lg_3 2}$, where $\epsilon(g)$ is a constant dependent on g.*

Theorem 28 (Sheppardson and Yu -02). *$\epsilon(g)$ can be replaced by a constant if G is also "locally planar".*

Fact 29. *3-connected nonplanar graphs other than K_5 are exactly those graphs containing $K_{3,3}$-minors.*

Conjecture 30 (Thomas). *For every positive integer t, there exist two positive real numbers α_t and β_t such that $c(G) \geq \alpha_t n^{\beta_t}$ for every 3-connected graph G on n vertices with no $K_{3,t}$-minor.*

Theorem 31 (C, Sheppardson, Yu, and Zang). *Let G be a 3-connected graph on n vertices and containing no $K_{3,t}$-minor. Then $c(G) \geq |G|^{r(t)}$, where $r(t) = \log_{8t^t+1} 2$.*

Theorem 32. *Let $r(t) := \log_{8t^t+1} 2$ and let G be a 3-connected graph with no $K_{3,t}$-minor. Then the following statements hold.*

(a) For any distinct vertices x, y, z of G such that $xz, yz \in E(G)$, $G-z$ contains an x-y path of length at least $(\frac{|G|-1}{t^\mu})^{r(t)}$, where $\mu := \mu(G; x, y, z)$.

(b) For any $xy \in E(G)$, G contains an x-y path of length at least $|G|^{r(t)}$.

(c) For any two distinct edges xy, f of G, G contains an x-y path through f which has length at least $(\frac{|G|}{t^t})^{r(t)} + 1$.

Proof Methodology

- Decompose 2-connected graphs into 3-connected components (Tutte's decomposition theorem).

- Individualize each edge by define ladder and rung.

- Correct errors by using interval graphs.

Conjecture 33 (Seymour and Thomas). *There is a universal constant* β *and function* $\alpha(t) > 0$ *such that* $c(G) \geq \alpha(t)n^{\beta}$ *for every 3-connected graph with no* $K_{3,t}$*-minor.*

Theorem 34 (C, Yu, and Zang, ??). *The above conjecture is true.*

Conjecture 35. *There is a function* $\alpha(t) > 0$ *such that* $c(G) \geq \alpha(t)n^{\log_3 2}$ *for every 3-connected graph with no* $K_{3,t}$*-minor.*

Conjecture 36. *There is a function* $\alpha(t) > 0$ *such that* $c(G) \geq \alpha(t)n$ *for every 6-connected graph with no* $K_{3,t}$*-minor.*

Question 37. *How about 4-connected or 5-connected graphs.*