VIZA 659 / CPSC 649

Addendum
Notes: 9/19/07

Derivation of RK2 or Midpoint Method for Numerical Integration

Taylor series for time step of $x(t)$

$$x(t+h) = x(t) + h\dot{x} + \frac{h^2}{2}\ddot{x} + \underbrace{O(h^3)}_{\text{estimate error - ignore}}$$

system dynamics $\dot{x} = f(x)$

by chain rule $\ddot{x} = \frac{d^2x}{dt^2} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} = f'f$

$$x(t+h) = x(t) + hf + \frac{h^2}{2}ff'$$

or

$$\Delta x = hf + \frac{h^2}{2}ff' \qquad (1)$$

Taylor series for system dynamics function

$$f(x+\delta) = f(x) + \delta f' + \underbrace{O(\delta^2)}_{\text{estimate error - ignore}}$$

Let δ be the midpoint of an Euler step

$$\delta = \frac{h}{2}f$$

$$f(x + \tfrac{h}{2}f) = f + \frac{h}{2}ff'$$

so

$$hf(x+\tfrac{h}{2}f) = hf + \frac{h^2}{2}ff' = \Delta x \quad \text{by eqn (1)}$$

Thus,

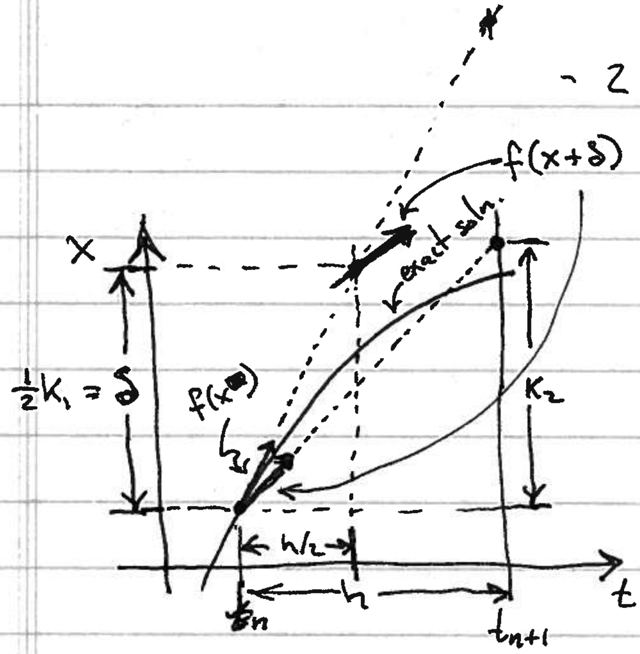
$$\Delta x = hf(x + \tfrac{h}{2}f)$$

RK2:

$$K_1 = hf(x^{(n)})$$

$$K_2 = hf(x^{(n)} + \tfrac{1}{2}K_1) \quad (\text{this is } \Delta x)$$

$$x^{(n+1)} = x^{(n)} + K_2$$



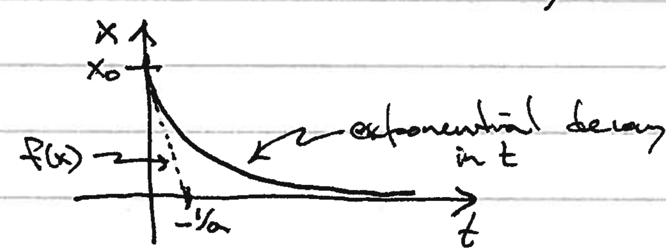
What are we doing with RK2 "Midpoint" method?

$\frac{1}{2}K_1$ is δ, i.e. it is the change in x at the midpoint of an Euler step, using $\dot{x} = f(x)$ as slope.

Now, find the slope at $x+\delta = x + \frac{1}{2}K_1$. It is $f(x+\frac{1}{2}K_1)$. Go back to start and take a full Euler step using this slope. This is $\Delta x = K_2 = h f(x+\frac{1}{2}K_1)$.

<u>Example: Exponential Decay</u>

$$\dot{x} = -ax \implies f(x) = -ax \quad , \text{ soln. } \quad x = x_0 e^{-at}$$



$$K_1 = hf(x) = -ahx$$
$$K_2 = hf(x+\tfrac{1}{2}K_1) = h\left[-a\left(x - \tfrac{ah}{2}x\right)\right]$$
$$= -ahx + \frac{a^2h^2}{2}x$$

$$x^{(n+1)} = x^{(n)} - ahx^{(n)} + \frac{a^2h^2}{2}x^{(n)}$$
$$x^{(n+1)} = \left(1 - ah + \frac{a^2h^2}{2}\right)x^{(n)} \quad , \text{ change quadratic in } h$$

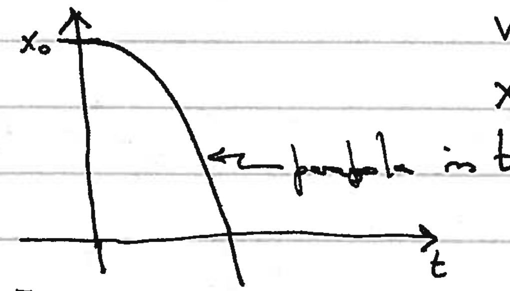
Euler would give

$$x^{(n+1)} = (1-ah)x^{(n)} \quad , \text{ change linear in } h$$

Example: Gravitational Acceleration

$$\underline{X} = \begin{bmatrix} x \\ v \end{bmatrix}, \quad f_v(\underline{X}) = -g, \quad f_x(\underline{X}) = v$$

$$\dot{\underline{X}} = \begin{bmatrix} v \\ -g \end{bmatrix} = f(\underline{X})$$



$$v = v_0 \cancel{+} - gt$$

$$x = x_0 + v_0 t - \tfrac{1}{2} g t^2$$

$$\underline{K}_1 = hf(\underline{X}) = h \begin{bmatrix} v \\ -g \end{bmatrix}$$

$$\underline{K}_2 = hf\left(\underline{X} + \tfrac{1}{2}\underline{K}_1\right) = \cancel{hf(\underline{X} + hv}$$

$$= hf\left(\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} \frac{1}{2}hv \\ -\frac{1}{2}hg \end{bmatrix}\right) = hf\left(\begin{bmatrix} x + \frac{1}{2}hv \\ v - \frac{1}{2}hg \end{bmatrix}\right)$$

$$= h \begin{bmatrix} v - \frac{1}{2}hg \\ -g \end{bmatrix}$$

$$\underline{X}^{(n+1)} = \underline{X}^{(n)} + \begin{bmatrix} hv - \frac{1}{2}h^2 g \\ -gh \end{bmatrix}$$

so: $x^{(n+1)} = x^{(n)} + hv^{(n)} - \frac{1}{2}h^2 g$, change parabolic in h

$v^{(n+1)} = v^{(n)} \cancel{+} - gh$, change linear in h

Note that solution is exact, no matter what the time step