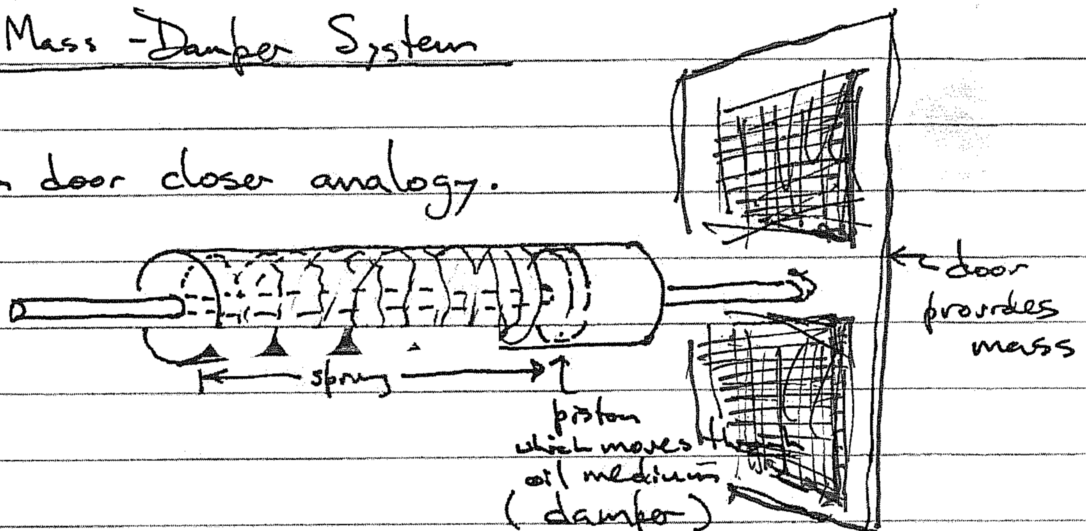
VIZA 659 / CPSC 649 Oct. 27, 2007

(Lecture recorded on DVD Oct. 24, 2007)

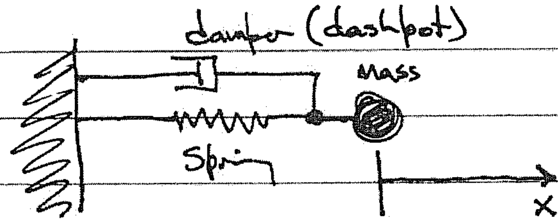
Spring - Mass - Damper System

Screen door closer analogy.



Spring provides force to close door - ~~compressed~~ stretched when door fully open. Pistons damps motion to constrain velocity. Doesn't cancel spring force, just slows rate at which it can be resolved.

1D Model



Pick origin so spring relaxed, spring force = 0 when $x = 0$

Spring force: $f_k = -kx$, linear spring force $\propto$ displa

Damper force: $f_d = -d\dot{x}$, linear damping force $\propto$ velocity
(dashpot)

External force: f_e

$$m\ddot{x} = f_e - kx - d\dot{x}$$

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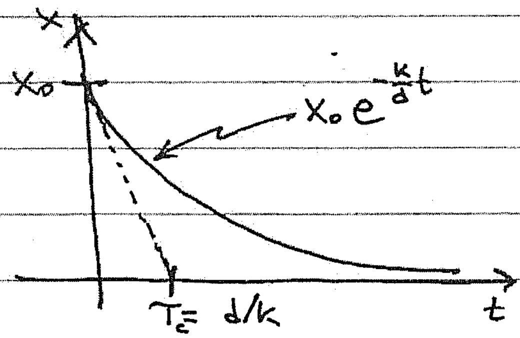
Let's see how this system acts without external force:

Massless Spring-Damper

$V = \dot{X} = -\frac{k}{d}X$, soln.. $X = X_0 e^{-\frac{k}{d}t}$

proof: $\dot{X} = -\frac{k}{d} X_0 e^{-\frac{k}{d}t} = -\frac{k}{d}X$



Characterized by "Time Constant"
T_c is ~~the~~ time it would take for system to go to 0 if initial slope (velocity) held constant

slope when t=0 $\quad \dot{X}\big|_{t=0} = -\frac{k}{d}X_0$

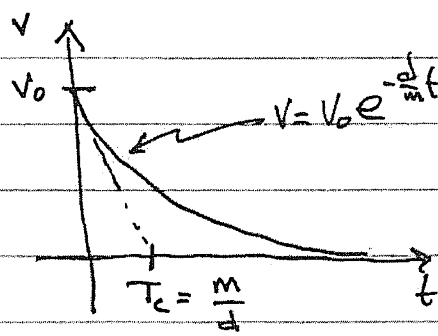
goes to 0 when $\Delta X = -X_0$

$$-\frac{k}{d}X_0 T_c = -X_0 \implies T_c = \frac{d}{k}$$

Springless Mass-Damper

$a = \ddot{X} = \dot{V} = -\frac{d}{m}V$, $\dot{V} = V_0 e^{-\frac{d}{m}t}$

proof: $\dot{V} = -\frac{d}{m}V_0 e^{-\frac{d}{m}t} = -\frac{d}{m}V$



"Time Constant" is $\frac{m}{d}$
by analogy to previous system

$T_c = m/d$

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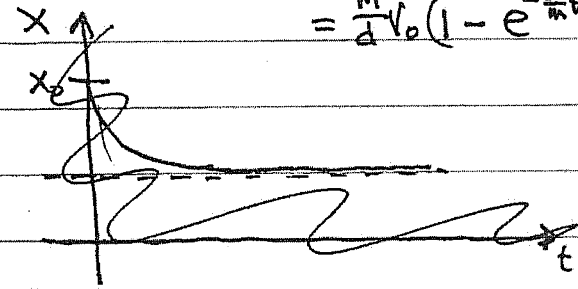
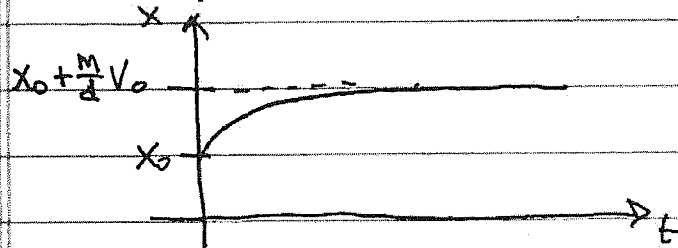
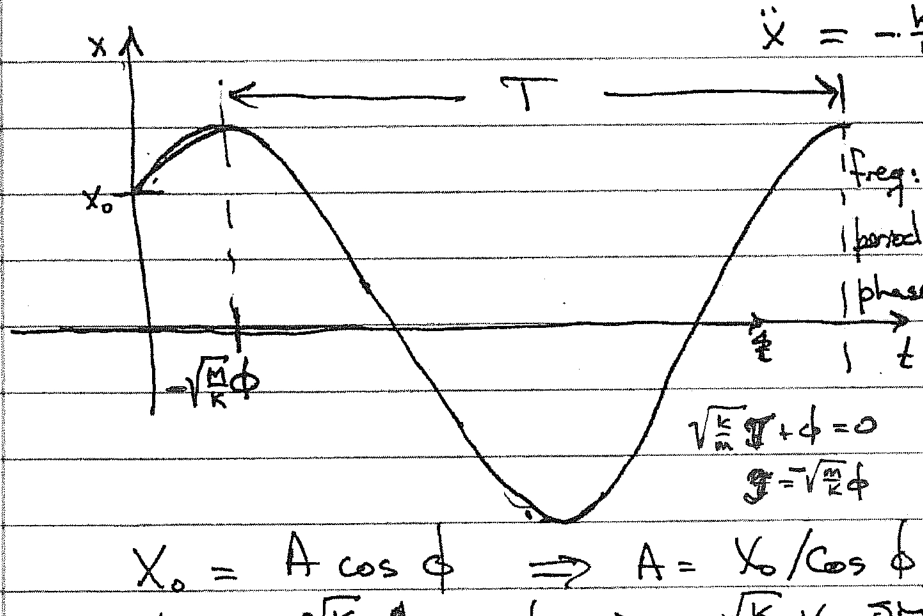
$$x = \int v\,dt = -\frac{m}{d}V_0\, e^{-\frac{d}{m}t} + \frac{m}{d}V_0 + x_0$$
$$= \frac{m}{d}V_0\left(1 - e^{-\frac{d}{m}t}\right) + x_0$$





Spring - Mass

$$a = \dot{v} = \ddot{x} = -\frac{k}{m}x \,, \qquad x = A\cos\left(\sqrt{\tfrac{k}{m}}\,t + \phi\right)$$
$$v = \dot{x} = -\sqrt{\tfrac{k}{m}}\,A\sin\left(\sqrt{\tfrac{k}{m}}\,t + \phi\right)$$
$$a = \ddot{x} = -\frac{k}{m}\,A\cos\left(\sqrt{\tfrac{k}{m}}\,t + \phi\right)$$
$$\ddot{x} = -\frac{k}{m}x$$



freq: $\omega = \sqrt{\frac{K}{m}}$

period: $T = 2\pi/\omega = 2\pi\sqrt{\frac{M}{K}}$

phase angle: ϕ

shift $= -\frac{\phi}{2\pi}T$

$= -\sqrt{\frac{M}{K}}\,\phi$

$\sqrt{\frac{k}{m}}\,t + \phi = 0$

$t = -\sqrt{\frac{m}{k}}\,\phi$

$$X_0 = A\cos\phi \implies A = X_0/\cos\phi$$
$$V_0 = -\sqrt{\tfrac{K}{m}}\,A\sin\phi \implies -\sqrt{\tfrac{K}{m}}\,X_0\,\frac{\sin\phi}{\cos\phi} = -\sqrt{\tfrac{K}{m}}\,X_0\tan\phi$$
$$\phi = \tan^{-1}\left(-\sqrt{\tfrac{m}{K}}\,\frac{V_0}{X_0}\right)$$

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Spring - Mass - Damper

$$m\ddot{x} + d\dot{x} + kx = 0$$

$$\ddot{x} + \frac{d}{m}\dot{x} + \frac{k}{m}x = 0$$

let $\omega_n = \sqrt{\frac{k}{m}}$, undamped natural frequency

$\zeta = \frac{d}{2\sqrt{km}}$, damping factor

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

assume solns. of form $\cancel{x} = e^{st}$

$\dot{x} = se^{st}$

$\ddot{x} = s^2e^{st}$

$$s^2e^{st} + 2\zeta\omega_n se^{st} + \omega_n^2 e^{st} = 0$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s = -\zeta\omega_n \pm \frac{1}{2}\sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}$$

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

if $\zeta < 1$ then

$$s = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$ (where $i = \sqrt{-1}$)

$$x = e^{-\zeta\omega_n t}\left(Ae^{i\omega\sqrt{1-\zeta^2}t} + Be^{-i\omega\sqrt{1-\zeta^2}t}\right)$$

$$x = Ce^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t + \phi\right)$$

where $C = \sqrt{A^2+B^2}$, $\phi = \tan^{-1} B/A$

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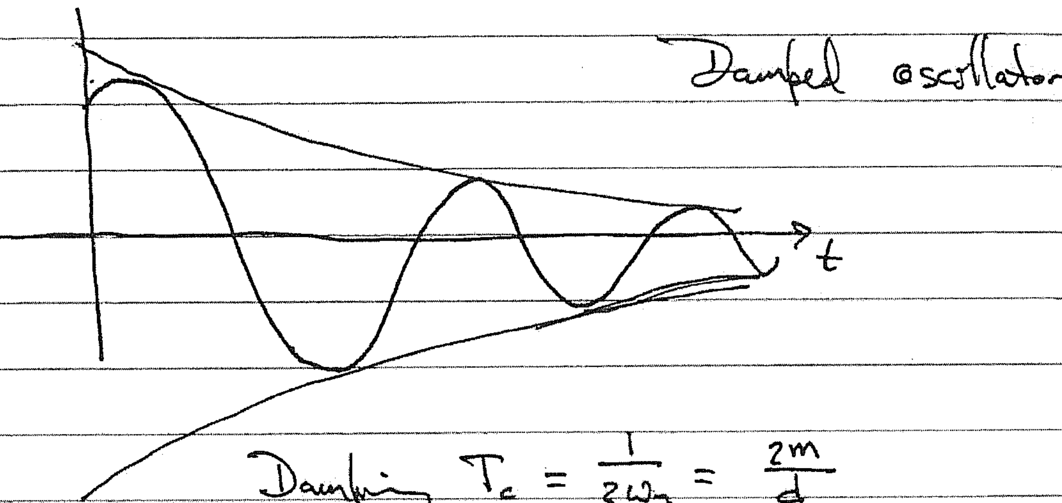
Behavior -



Damping $T_c = \frac{1}{\zeta\omega_n} = \frac{2m}{d}$

~~Period~~ $\cancel{T \approx 2\pi\sqrt{\frac{m}{k}}}$ ~~for small~~ $\cancel{\zeta}$

$\cancel{= 2\pi\sqrt{(1-\zeta^2)\frac{m}{k}}}$ ~~exactly~~

$\cancel{= 2\pi\sqrt{\frac{m}{(1-\zeta^2)k}}}$

Frequency $\omega = \sqrt{1-\zeta^2}\,\omega_n$, $\omega_n = \sqrt{k/m}$

for small ζ, $\omega \approx \omega_n$

Period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{(1-\zeta^2)k}}$

for small ζ, $T \approx 2\pi\sqrt{\frac{m}{k}}$